應用數學

First order differential eq.（一階微分方程）

Differential eq.：含未知函數及其導數的方程式

Ordinary diff. eq.（常微分方程式）：未知函數僅有一個變數

1. $\ddot{y}+\omega^2 y=0$ ， $y=y(t)$（y 上面的兩點指的是對時間 t 微分）
2. $y'=e^x$ ， $y=y(x)$
3. $x^2y''+xy'+y=0$ ， $y=y(x)$

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Partial diff. eq.（偏微分方程式）：未知函數有 2 個 or 2 個以上變數。

1. $\nabla^2 V=0$（Laplace's eq.）

   $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ ， $V=V(x,y,z)$
2. $\frac{1}{C^2}\frac{\partial^2}{\partial t^2}U-\frac{\partial^2}{\partial x^2}U=0$ ，（wave eq.）

   $U=U(t,x)$
3. $\frac{\partial U}{\partial t}=C^2\frac{\partial U}{\partial x}$ ，（heat eq.）

   $U=U(t,x)$
4. $i\hbar\frac{\partial\varphi}{\partial t}=\frac{\hbar^2}{2m}\nabla^2\varphi+V(x,y,z)\varphi$ ，（schrödinger eq.）

   $\varphi=\varphi(t,x,y,z)$

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Order（階）：微分方程式中，出現最高導數的階數。

Degree（次）：微分方程式中，有理化後最高階導數的次數。

1. $y'=\sin x$ 　　一階一次
2. $y''+y=\cos x$ 　　二階一次
3. $y'''+\sqrt{y'}+x^2y=0$ 　　三階二次

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Linear diff. eq.：微分方程中，未知函數及其各階導數均為一次，豈無彼此互乘著。

1. $y'+(\cos x)y=e^x$
2. $xy''+y'=x^2$
3. $xy''-e^xy'+(\sin x)y=0$

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線性微分一般形式： $a_n(x)y^n+a_{n-1}(x)y^{n-1}+......+a_1(x)y'+a_0(x)y=f(x)$

$$\begin{cases} f(x+y)=f(x)+f(y) \\ f(cx)=cf(x) \end{cases}$$

$f(x)$ ：線性函數。c：常數。

$Oy = f(x)$　　　O：diff. operator（算符）

$$O = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{(n-1)}}{dx^{(n-1)}} + \dots\dots + a_1(x)\frac{d}{dx} + a_0(x)$$

NonLinear diff. eq.：非線性微分方程式

1. $y' + y^2 = 1$
2. $y'' + yy' = \sin x$
3. $y''' + (y')^2 + y = 0$

First order diff. eq.

1.separable： $g(y)y' = f(x) \Rightarrow g(y)dy = f(x)dx$

① $y' = 2xy^2 \Rightarrow \dfrac{dy}{dx} = 2xy^2 \Rightarrow \dfrac{1}{y^2}dy = 2xdx$

$\Rightarrow -\dfrac{1}{y} = x^2 + c$（c：積分常數）

$\Rightarrow y = -\dfrac{1}{x^2 + c}$ …#

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② $y' = -ky \Rightarrow \dfrac{dy}{dt} = -ky \Rightarrow \dfrac{1}{y}dy = -kdt$

$\Rightarrow \ln y = -kt + c$

$\Rightarrow y = ce^{-kt}$

$\Rightarrow$ if $y(0) = y_0$

$\Rightarrow y = y_0 e^{-kt}$

If $y = y_0 2^{-\frac{t}{T}} \Rightarrow e^{-kt} = 2^{-\frac{t}{T}}$

$\Rightarrow -kt = -\dfrac{t}{T}\ln 2$

$\Rightarrow T = \dfrac{\ln 2}{k}$（半衰期）

If $t = T$，$y = y_0 2^{-1} \Rightarrow y = \dfrac{1}{2}y_0$

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③ $y' = \dfrac{x}{y}e^{(x^2+y^2)} \Rightarrow ye^{-y^2}dy = xe^{x^2}dx \Rightarrow -\dfrac{1}{2}e^{-y} = \dfrac{1}{2}e^{x^2} + c$

$\Rightarrow e^{x^2} + e^{y^2} = c$

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④ $L\dfrac{dI}{dt} + RI = 0 \Rightarrow \dfrac{dI}{dt} + \dfrac{RI}{L} = 0 \Rightarrow \dfrac{1}{I}dI = -\dfrac{R}{L}dt$

$\Rightarrow \ln I = -\dfrac{R}{L}t + c$

$$\Rightarrow \; I = I_0 e^{-\frac{R}{L}t} \quad (I_0 = e^c)$$

⑤ $y' = \dfrac{\cos x}{3y^2 + e^y}$ ， $y(0) = 2$ ：initial condition

$(3y^2 + e^y)dy = \cos x dx \;\Rightarrow\; y^3 + e^y = \sin x + c$

$\Rightarrow\; 8 + e^2 = c$

$\Rightarrow\; y^3 + e^y = \sin x + e^2 + 8$

雙曲

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\begin{cases} \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots \\ \sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \ldots \end{cases}$$

$$\begin{cases} \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots \\ \cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots \end{cases}$$

$\int \dfrac{dt}{\sqrt{1-t^2}} = \sin^{-1} t$ ， $\int \dfrac{dt}{\sqrt{1+t^2}} = \sinh^{-1} t$

☆ $\int \dfrac{dv}{1-v^2} = \tanh^{-1} v + c = \dfrac{1}{2}\int (\dfrac{1}{1+v} + \dfrac{1}{1-v})dv = \dfrac{1}{2}\ln\dfrac{1+v}{1-v}$

⑥ $m\dfrac{dv}{dt} = mg - \alpha v^2 \;\Rightarrow\; \dfrac{dv}{dt} = g - \dfrac{\alpha}{m}v^2 = g(1 - \dfrac{\alpha}{mg}v^2) \;\Rightarrow\; \dfrac{dv}{1 - \dfrac{\alpha}{mg}v^2} = gdt$

$\Rightarrow$ Let $S^2 = \dfrac{\alpha}{mg}v^2 \;\Rightarrow\; S = \sqrt{\dfrac{\alpha}{mg}}v \;\Rightarrow\; dS = \sqrt{\dfrac{\alpha}{mg}}dv$

$\Rightarrow\; \sqrt{\dfrac{mg}{\alpha}}\int \dfrac{dS}{1-S^2} = gt + c$

$\Rightarrow\; \tanh^{-1} S = \sqrt{\dfrac{\alpha g}{m}}t + c \;\Rightarrow\; \tanh^{-1}\sqrt{\dfrac{\alpha}{mg}}v = \sqrt{\dfrac{\alpha g}{m}}t + c$

$\Rightarrow\; v = \sqrt{\dfrac{mg}{\alpha}}\tanh(\sqrt{\dfrac{\alpha g}{m}}t + c)$

$\Rightarrow\; t \to \infty$ ， $V(\infty)$ ：terminal velocity 終端速度

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \to 1$

⑦ $\dfrac{dy}{dx} = xe^x y^2 \;\Rightarrow\; \dfrac{dy}{y^2} = xe^x dx \;\Rightarrow\; -\dfrac{1}{y} = xe^x - \int e^x dx$

$\Rightarrow\; -\dfrac{1}{y} = xe^x - e^x + c$

$$\Rightarrow \ y=\frac{1}{e^{x}-xe^{x}-c}$$

⑧ $\frac{dy}{dx}=ay-by^{2}$，$y(0)=\frac{a}{a+b}$，求 $\lim_{x\to\infty} y(x)$

$$\frac{1}{y-\frac{b}{a}y^{2}}dy=adx \ \Rightarrow \ (\frac{A}{y}-\frac{B}{1-\frac{b}{a}y})dy=adx$$

$$\Rightarrow \text{Let } \frac{b}{a}=c \ \Rightarrow \ (\frac{A}{y}-\frac{B}{1-cy})dy=adx \ \Rightarrow \ A=1\text{，}B=-c$$

$$\Rightarrow \ (\frac{1}{y}+\frac{\frac{b}{a}}{1-\frac{b}{a}y})dy=adx$$

$$\Rightarrow \ \ln y-\ln(1-\frac{b}{a}y)=ax+c$$

$$\Rightarrow \ \frac{y}{1-\frac{b}{a}y}=ce^{ax}\text{（}y(0)=\frac{a}{a+b}\text{代入上式）}\Rightarrow \frac{y}{1-\frac{b}{a}y}=e^{ax}$$

$$\Rightarrow \ y=e^{ax}-\frac{b}{a}e^{ax}y$$

$$\Rightarrow \ y(1+\frac{b}{a}e^{ax})=e^{ax}$$

$$\Rightarrow \ y=\frac{e^{ax}}{1+\frac{b}{a}e^{ax}}=\frac{1}{\frac{1}{e^{ax}}+\frac{b}{a}}$$

$$\Rightarrow \ \lim_{x\to\infty} y(x)=\lim_{x\to\infty}\frac{1}{\frac{1}{e^{ax}}+\frac{b}{a}}=\frac{a}{b}$$

2.Reduction to separable form

① $y'=(x+y)^{2}$

$$\Rightarrow \text{Let } \ x+y=t \ \Rightarrow \ dx+dy=dt \ \Rightarrow \ \frac{dy}{dx}=\frac{dt}{dx}-1$$

$$\Rightarrow \frac{dt}{dx}-1=t^{2} \ \Rightarrow \ \frac{dt}{1+t^{2}}=dx$$

$$\Rightarrow \ \tan^{-1}t=x+c$$

$$\Rightarrow \ y=\tan(x+c)-x$$

② $(2x-4y+5)y'+(x-2y+3)=0$

$$\Rightarrow \text{Let } \ x-2y=t \ \Rightarrow \ \frac{1}{2}(1-\frac{dt}{dx})=\frac{dy}{dx}$$

$$\Rightarrow \ (2t+5)\frac{1}{2}(1-\frac{dt}{dx})+t+3=0$$

$$\Rightarrow 1-\frac{dt}{dx}=-\frac{2(t+3)}{2t+5}$$

$$\Rightarrow \frac{dt}{dx}=\frac{4t+11}{2t+5} \Rightarrow \frac{2t+5}{4t+11}dt=dx$$

$$\Rightarrow (\frac{1}{2}-\frac{\frac{1}{2}}{4t+11})dt=dx$$

$$\Rightarrow \frac{t}{2}-\frac{1}{8}\ln(4t+11)=x$$

$$\Rightarrow \frac{x-2y}{2}-\frac{1}{8}\ln(4x-8y+11)=x+c$$

$$\Rightarrow 4x+8y+\ln(4x-8y+11)=c$$

應用數學

Homogeneous function

$f(\lambda x,\lambda y,......)=\lambda^r f(x,y,......)$ 稱爲 r 次的齊次函數

③ $2xyy'-y^2+x^2=0$

$$\Rightarrow \text{Let } u=\frac{y}{x} \Rightarrow y=ux \Rightarrow dy=udx+xdu \Rightarrow \frac{dy}{dx}=u+x\frac{du}{dx}$$

$$\Rightarrow 2ux^2(u+x\frac{du}{dx})-u^2x^2+x^2=0$$

$$\Rightarrow 2u(u+x\frac{du}{dx})-u^2+1=0$$

$$\Rightarrow u^2+1+2ux\frac{du}{dx}=0 \Rightarrow 2ux\frac{du}{dx}=-(u^2+1)$$

$$\Rightarrow -\frac{2udu}{u^2+1}=\frac{1}{x}dx \Rightarrow -\ln(u^2+1)=\ln x+c$$

$$\Rightarrow \frac{1}{u^2+1}=cx$$

$$\Rightarrow x^2+y^2=cx$$

④ $xy'=\frac{y^2}{x}+y \Rightarrow y'=(\frac{y}{x})^2+(\frac{y}{x})$

$$\Rightarrow \text{Let } \frac{y}{x}=t \Rightarrow y=xt \Rightarrow dy=xdt+tdx \Rightarrow \frac{dy}{dx}=x\frac{dt}{dx}+t$$

$$\Rightarrow x\frac{dt}{dx}+t=t^2+t$$

$$\Rightarrow \frac{1}{t^2}dt=\frac{1}{x}dx$$

$$\Rightarrow -\frac{1}{t}=\ln x+c \Rightarrow -\frac{x}{y}=\ln x+c$$

$$\Rightarrow y=-\frac{x}{\ln x+c}$$

⑤ $y' = \dfrac{2y - x + 5}{2x - y - 4}$ （想法子消去常數項，才可以用齊次）

$$\Rightarrow \text{Let} \begin{cases} u = x - \alpha \\ v = y - \beta \end{cases} \Rightarrow \begin{cases} x = u + \alpha \\ y = v + \beta \end{cases} \Rightarrow \begin{cases} dx = du \\ dy = dv \end{cases}$$

$$\Rightarrow \frac{dv}{du} = \frac{2v + 2\beta - u - \alpha + 5}{2u + 2\alpha - v - \beta - 4}$$

$$\Rightarrow \begin{cases} -\alpha + 2\beta + 5 = 0 \\ 2\alpha - \beta - 4 = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = -2 \end{cases}$$

$$\Rightarrow \frac{dv}{du} = \frac{2v - u}{2u - v}$$

$$\Rightarrow \text{Let } \frac{v}{u} = t \Rightarrow v = ut \Rightarrow dv = tdu + udt \Rightarrow \frac{dv}{du} = t + u\frac{dt}{du}$$

$$\Rightarrow t + u\frac{dt}{du} = \frac{2ut - u}{2u - ut} = \frac{2t - 1}{2 - t}$$

$$\Rightarrow y - x + 3 = c(y + x + 1)^3$$

3.Exact（正合）

$M(x, y)dx + N(x, y)dy = 0$

$\Rightarrow du = 0$ ， $u = u(x, y)$

$$du = (\frac{\partial u}{\partial x})dx + (\frac{\partial u}{\partial y})dy \qquad [\, M = \frac{\partial u}{\partial x} \,,\, N = \frac{\partial u}{\partial y} \,]$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \,,\, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Leftrightarrow Mdx + Ndy$ 爲 exact 充要條件

① $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^3 + 3xy^2) = 6xy \,,\, \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3x^2y + y^3) = 6xy$$

$$u = \int (x^3 + 3xy^2)dx = \frac{x^4}{4} + \frac{3}{2}x^2y^2 + f(y)$$

$$N = \frac{\partial u}{\partial y} = 3x^2y + f'(y) \Rightarrow f'(y) = y^3 \Rightarrow f(y) = \frac{1}{4}y^4 + c$$

$$u = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + c' = c''$$

$$\Rightarrow x^4 + 6x^2y^2 + y^4 = c$$

② $xe^{xy}y' + ye^{xy} - 4x^3 = 0 \Rightarrow xe^{xy}dy + (ye^{xy} - 4x^3)dx = 0$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(xe^{xy})=e^{xy}+xye^{xy}\ ,\ \frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(ye^{xy}-4x^3)=e^{xy}+xye^{xy}$$

$$u=\int xe^{xy}dy=e^{xy}+f(x)$$

$$M=\frac{\partial u}{\partial x}=ye^{xy}+f'(x)$$

$$f'(x)=-4x^3\ \Rightarrow\ f(x)=-x^4$$

$$u=e^{xy}-x^4=c$$

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③ $[\sin y-y\sin(xy)]dx-[x\cos y-x\sin(xy)]dy=0$

$$\frac{\partial M}{\partial y}=\cos y-\sin(xy)-xy\cos(xy)\ ,\ \frac{\partial N}{\partial x}=\cos y-\sin(xy)-xy\cos(xy)$$

$$u=\int[\sin y-y\sin(xy)]dx=x\sin y+\cos(xy)+f(y)$$

$$N=\frac{\partial u}{\partial y}=x\cos y-x\sin(xy)+f'(y)$$

$$f'(y)=0$$

$$\therefore\ f(y)=c$$

$$u=x\sin y+\cos(xy)+c$$

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④ $ydx-xdy=0$

$\frac{\partial M}{\partial y}=1\neq\frac{\partial N}{\partial x}=-1$（非正合型）（非 Exact）

Integrating factor：乘上積分因子後，方程式才會變成正合型

$P(x,y)dx+Q(x,y)dy\ \Rightarrow\ F\cdot P(x,y)dx+F\cdot Q(x,y)dy$（$F$ 為積分因子）

$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(FP)=\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(FQ)\ \Rightarrow$ Exact

$F_yP+FP_y=F_xQ+FQ_x$（要解 F，一般而言會很複雜）

若 $F=F(x)\ \Rightarrow\ FP_y=QF_x+FQ_x\ \Rightarrow\ \frac{1}{F}\frac{dF}{dx}=\frac{1}{Q}(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})$

若 $F=F(y)\ \Rightarrow\ F_yP+FP_y=FQ_x\ \Rightarrow\ \frac{1}{F}\frac{dF}{dy}=\frac{1}{P}(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y})$

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⑤ $2xydx+(4y+3x^2)dy=0$

$$\begin{cases} M=F(2xy) \\ N=F(4y+3x^2) \end{cases}\Rightarrow\begin{cases} \frac{\partial M}{\partial y}=F'(2xy)+F(2x) \\ \frac{\partial N}{\partial x}=F(6x) \end{cases}$$

$$F'(2xy)=F(4x) \Rightarrow yF'=2F \Rightarrow \frac{1}{F}dF=\frac{2}{y}dy \Rightarrow \ln F=2\ln y \Rightarrow F=y^2$$

原式 $\Rightarrow 2xy^3dx+(4y^3+3x^2y^2)dy=0$

$$u=\int 2xy^3dx=x^2y^3+f(y)$$

$$N=\frac{\partial N}{\partial y}=3x^2y^2+f'(y) \Rightarrow f'(y)=4y^3 \Rightarrow f(y)=y^4+c$$

$$u=x^2y^3+y^4+c$$

應用數學

4.全微分型（觀察法）

$$ydx+xdy=d(xy)$$

$$xdx+ydy=d[\frac{1}{2}(x^2+y^2)]$$

$$\frac{xdy-ydx}{xy}=d\ln(\frac{y}{x})$$

$$mydx+nxdy=\frac{d(x^my^n)}{x^{m-1}y^{n-1}}$$

$$\frac{xdy-ydx}{x^2}=d(\frac{y}{x})$$

$$\frac{xdy-ydx}{x^2+y^2}=d(\tan^{-1}\frac{y}{x})$$

$$\frac{xdy-ydx}{y^2}=-d(\frac{x}{y})$$

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① $(x-4x^2y^3)y'+3x^4-y=0 \Rightarrow (x-4x^2y^3)dy+(3x^4-y)dx=0$

$\Rightarrow xdy-ydx+x^2(-4y^3dy+3x^2dx)=0$

$\Rightarrow d(\frac{y}{x})-d(y^4)+d(x^3)=0$

$\Rightarrow d(\frac{y}{x}-y^4+x^3)=0$

$\Rightarrow \frac{y}{x}-y^4+x^3=c$

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② $(x-2x^2y-2y^3)y'-y=0 \Rightarrow (x-2x^2y-2y^3)dy-ydx=0$

$\Rightarrow xdy-ydx-2y(x^2+y^2)dy=0$

$\Rightarrow d(\tan^{-1}\frac{y}{x})-d(y^2)=0$

$$\Rightarrow\ \tan^{-1}\frac{y}{x} - y^2 = c$$

③ $yy' + x - x^2 - y^2 = 0 \Rightarrow ydy + xdx - (x^2+y^2)dx = 0$

$$\Rightarrow \frac{\frac{1}{2}d(x^2+y^2)}{x^2+y^2} = dx$$

$$\Rightarrow \frac{1}{2}d\ln(x^2+y^2) = dx$$

$$\Rightarrow \ln(x^2+y^2) - 2x = c$$

$$\Rightarrow x^2 + y^2 = ce^{2x}$$

5.First Order Linear Equation（一階線性微分方程）

$y' + p(x)y = r(x)$

$r(x) = 0$……homogeneous

$r(x) \neq 0$……nonhomogeneous

$r(x) = 0$ $\quad y = y_h$

$$y_h' + p(x)y_h = 0$$

$$\frac{dy_h}{y_h} + p(x)dx = 0$$

$$\ln y_h + \int p(x)dx = 0 \Rightarrow y_h = e^{-\int p(x)dx}$$

General solution 通解

$r(x) \neq 0$ $\quad y(x) = y_h(x)V(x)$

$y' = y_h'V + y_hV'$ 代入原式 $y' + p(x)y = r(x)$ 得

$y_h'V + y_hV' + py_hV = r$

其中 $y_h' + p(x)y_h = 0$

所以 $y_hV' = r$

$$V' = \frac{r}{y_h} \Rightarrow V = \int \frac{r(x)}{y_h(x)}dx$$

$$\boxed{y = e^{-\int p(x)dx}\left[\int r(x)e^{\int p(x)dx} + c\right]}$$ ……公式

① $y' - y = e^{2x} \Rightarrow y_h' - y_h = 0$

$$\Rightarrow y_h = e^{-\int(-1)dx} = e^x$$

$$\Rightarrow y = y_hV$$

$$\Rightarrow y_h'V + y_hV' - y_hV = e^{2x}\ (\ y_h' - y_h = 0\ )$$

$$\Rightarrow V' = e^x$$

$$\Rightarrow\ V = e^{x} + c$$
$$\Rightarrow\ y = e^{2x} + ce^{x}$$

② $(x+1)y' - y = x \ \Rightarrow\ y' - \dfrac{y}{x+1} = \dfrac{x}{x+1}$

$$\Rightarrow\ y_h = e^{\int \frac{1}{x+1}dx} = e^{\ln(x+1)} = x+1$$
$$\Rightarrow\ y = y_h V$$
$$\Rightarrow\ y_h'V + y_hV' - \frac{1}{x+1}y_hV = \frac{x}{x+1}$$
$$\Rightarrow\ V' = \frac{x}{y_h(x+1)} = \frac{x}{(x+1)^2}$$
$$\Rightarrow\ V = \int \frac{x}{(x+1)^2}dx = \int [\frac{1}{x+1} - \frac{1}{(x+1)^2}]dx$$
$$= \ln(x+1) + \frac{1}{x+1} + c$$
$$\Rightarrow\ y = c(x+1) + 1 + (x+1)\ln(x+1)$$

③ $y' + (\tan x)y = \sin 2x \ \Rightarrow\ y_h = e^{-\int \tan x dx} = e^{\ln\cos x} = \cos x$

$$\Rightarrow\ y = (\cos x)V$$
$$\Rightarrow\ y' = -\sin xV + \cos xV'$$
$$\Rightarrow\ (-\sin x)V + (\cos x)V' + (\tan x)(\cos x)V = \sin 2x$$
$$\Rightarrow\ V' = 2\sin x \ \Rightarrow\ V = -2\cos x + c$$
$$\Rightarrow\ y = c\cos x - 2\cos^2 x$$

④ $y' - \dfrac{2}{x}y = x^2\cos 3x \ \Rightarrow\ y_h = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$$\Rightarrow\ y = x^2V$$
$$\Rightarrow\ y' = 2xV + x^2V'$$
$$\Rightarrow\ 2xV + x^2V' - 2xV = x^2\cos 3x$$
$$\Rightarrow\ V' = \cos 3x \ \Rightarrow\ V = \frac{1}{3}\sin 3x + c$$
$$\Rightarrow\ y = \frac{x^3}{3}\sin 3x + cx^2$$

⑤ $y' - 2xy = e^{x^2} \ \Rightarrow\ y_h = e^{\int 2xdx} = e^{x^2}$

$$\Rightarrow\ y = e^{x^2}V \ \Rightarrow\ y' = 2xe^{x^2}V + e^{x^2}V'$$
$$\Rightarrow\ 2xe^{x^2}V + e^{x^2}V' - 2xe^{x^2}V = e^{x^2}$$
$$\Rightarrow\ V' = 1 \ \Rightarrow\ V = x + c$$
$$\Rightarrow\ y = xe^{x^2} + ce^{x^2}$$

反超越函數：$\sinh^{-1}x=\ln(x+\sqrt{x^2+1})$，for all x

$\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$，$x\ge 1$

$\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{1+x^2}}$，for all x

$\frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^2-1}}$，$x\ge 1$

$\tanh^{-1}x=\frac{1}{2}\ln\frac{1+x}{1-x}$，$-1<x<1$

$\coth^{-1}x=\frac{1}{2}\ln\frac{1+x}{1-x}$，$|x|>1$

$\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2}$，$|x|<1$

$\frac{d}{dx}\coth^{-1}x=\frac{1}{1-x^2}$，$|x|>1$

6.Reduction to Linear form Bernoulli equation（白努力方程式）

$$\frac{dy}{dx}+p(x)y=g(x)y^a$$

$$y^{-a}\frac{dy}{dx}+p(x)y^{1-a}=g(x)$$

Let $u=y^{1-a}$，$du=(1-a)y^{-a}dy$

$$\frac{1}{1-a}\frac{du}{dx}+p(x)u=g(x)$$

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① $y'+\frac{3}{x}y=x^2y^2y \Rightarrow y^{-2}y'+\frac{3}{x}y^{-1}=x^2$

$\Rightarrow$ Let $u=y^{-1}$，$du=-y^{-2}dy$

$\Rightarrow -\frac{du}{dx}+\frac{3}{x}u=x^2$

$\Rightarrow \frac{du}{dx}-\frac{3}{x}u=-x^2$

$\Rightarrow u_h=e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3$

$\Rightarrow u=x^3V$

$\Rightarrow du=3x^2Vdx+x^3dV$

$\Rightarrow 3x^2V+x^3\frac{dV}{dx}-\frac{3}{x}x^3V=-x^2$

$\Rightarrow \frac{dV}{dx}=-\frac{1}{x} \Rightarrow dV=-\frac{1}{x}dx \Rightarrow V=-\ln x+c$

$\Rightarrow u=y^{-1}=x^3V=x^3(-\ln x+c)$

$$\Rightarrow\ y=\frac{1}{x^3(-\ln x+c)}$$

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② $y'+\frac{1}{x}y=3x^2y^3\mathrm{y}\Rightarrow\ y^{-3}y'+\frac{1}{x}y^{-2}=3x^2$

$\Rightarrow$ Let $t=y^{-2}$ ， $t'=-2y^{-3}y'$

$$\Rightarrow\ -\frac{1}{2}t'+\frac{1}{x}t=3x^2$$

$$\Rightarrow\ t'-\frac{2}{x}t=-6x^2$$

$$\Rightarrow\ t_h=e^{\int\frac{2}{x}dx}=x^2$$

$$\Rightarrow\ t=t_hV\ \ \Rightarrow\ t'=t_h'V+t_hV'$$

$$\Rightarrow\ t_h'V+t_hV'-\frac{2}{x}t_hV=-6x^2$$

$$\Rightarrow\ x^2V'=-6x^2\ \Rightarrow\ V'=-6\ \Rightarrow\ V=-6x+c$$

$$\Rightarrow\ t=-6x^3+x-cx^2\ (\ t=y^{-2}\ )$$

$$\Rightarrow y=(cx^2-6x^3)^{-\frac{1}{2}}$$

Ricatti equation

$\frac{dy}{dx}=P(x)+Q(x)y+R(x)y^2$ ，若是已知 $y_1(x)$ 爲一特解

$$\Rightarrow\ y=y_1+u$$

$$\Rightarrow\ \frac{d}{dx}(y_1+u)=P+Q(y_1+u)+R(y_1+u)^2$$

$$\Rightarrow\ \frac{du}{dx}=Qu+2Ry_1u+Ru^2=(Q+2Ry_1)u+Ru^2$$

$$\Rightarrow\ \frac{du}{dx}-(Q+2Ry_1)u=Ru^2$$

---

③ $y'=\frac{1}{x}y^2+\frac{1}{x}y-\frac{2}{x}$ ， $y=1$ 爲特解（？？）

Let $y=1+u$ ， $y'=u'$

$$\Rightarrow\ u'=\frac{1}{x}(1+u)^2+\frac{1}{x}(1+u)-\frac{2}{x}=\frac{2}{x}u+\frac{1}{x}u^2+\frac{1}{x}u$$

$$\Rightarrow\ u'-\frac{3}{x}u=\frac{1}{x}u^2$$

$$\Rightarrow\ u^{-2}u'-\frac{3}{x}u^{-1}=\frac{1}{x}$$

Let $g=u^{-1}$ ， $dg=-u^{-2}du$

$$\Rightarrow\ -\frac{dg}{dx}-\frac{3}{x}g=\frac{1}{x}\ \Rightarrow\ \frac{dg}{dx}+\frac{3}{x}g=-\frac{1}{x}$$

$$\Rightarrow g_h = e^{-\int \frac{3}{x}dx} = \frac{1}{x^3} \Rightarrow g = \frac{1}{x^3}V$$

$$\Rightarrow dg = -3x^{-4}dxV + \frac{1}{x^3}dV$$ （代入 $\frac{dg}{dx} + \frac{3}{x}g = -\frac{1}{x}$ 得）

$$\Rightarrow -3x^{-4}V + \frac{1}{x^3}\frac{dV}{dx} + \frac{3}{x}(\frac{1}{x^3}V) = -\frac{1}{x}$$

$$\Rightarrow \frac{dV}{dx} = -x^2 \Rightarrow V = -\frac{1}{3}x^3 + c$$ （代回 $g = \frac{1}{x^3}V$ 得）

$$\Rightarrow g = -\frac{1}{3} + c'\frac{1}{x^3}$$

$$\Rightarrow u = g^{-1} = [-\frac{1}{3} + \frac{c'}{x^3}]^{-1}$$

$$\Rightarrow y = 1 + u = 1 + \frac{1}{\frac{-1}{3} + \frac{c'}{x^3}} = 1 + \frac{3x^3}{-x^3 + 3c} = \frac{2x^3 + 3c}{-x^3 + 3c}$$

---

④ $y' + y^2 - \frac{2}{x}y + \frac{2}{x^2} = 0$，$y = \frac{1}{x}$ 是一特解

Let $y = \frac{1}{x} + u \Rightarrow y' = -\frac{1}{x^2} + u'$

$$\Rightarrow -\frac{1}{x^2} + u' + (\frac{1}{x} + u)^2 - \frac{2}{x}(\frac{1}{x} + u) + \frac{2}{x^2} = 0$$

$$\Rightarrow -\frac{1}{x^2} + u' + \frac{1}{x^2} + \frac{2u}{x} + u^2 - \frac{2}{x^2} - \frac{2u}{x} + \frac{2}{x^2} = 0$$

$$\Rightarrow u' = -u^2 \Rightarrow \frac{du}{dx} = -u^2 \Rightarrow \frac{du}{-u^2} = dx \Rightarrow \frac{1}{u} = x + c \Rightarrow u = \frac{1}{x+c}$$

$$\Rightarrow \frac{1}{x+c} = y - \frac{1}{x} \Rightarrow y = \frac{1}{x} + \frac{1}{x+c}$$

---

⑤ $y' = x^2 - 2xy + y^2$，$y = x + 1$ 爲一特解

$$\Rightarrow y' = (x-y)^2 \Rightarrow \text{Let } y = x + 1 + u \Rightarrow y' = 1 + u'$$

$$\Rightarrow 1 + u' = (x - x - 1 - u)^2 = 1 + 2u + u^2$$

$$\Rightarrow \frac{1}{u(u+2)}du = dx \Rightarrow \frac{1}{2}(\frac{1}{u} - \frac{1}{u+2})du = dx$$

$$\Rightarrow \ln u - \ln(u+2) = 2x + c$$

$$\Rightarrow \frac{u}{u+2} = ce^{2x} \Rightarrow u = (u+2)e^{2x} \Rightarrow -2ce^{2x} = u(ce^{2x} - 1)$$

$$\Rightarrow u = \frac{-2ce^{2x}}{ce^{2x} - 1} \Rightarrow y - (x+1) = \frac{-2ce^{2x}}{ce^{2x} - 1}$$

$$\Rightarrow y = x + 1 + \frac{-2ce^{2x}}{ce^{2x} - 1}$$

Clairaut's equation

$$y = xy' + f(y')$$

微分 $\Rightarrow$ $y' = y' + xy'' + f'(y')y'$ （ $f'(y') = \frac{d}{dy'} f(y')$ ）

$$y''[x + f'(y')] = 0$$

Let $y'' = 0 \Rightarrow y' = c \Rightarrow y = cx + f(c)$

$$x + f'(y') = 0$$

………

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⑥ $y = xy' + (y')^2 \Rightarrow y' = y' + xy'' + 2y'y''$

$\Rightarrow y''(x + 2y') = 0$

$\Rightarrow$ (1) $y'' = 0 \Rightarrow y' = c \Rightarrow y = cx + c'$

(2) $x + 2y' = 0 \Rightarrow 2\frac{dy}{dx} = -x$

$\Rightarrow 2dy = -xdx$

$\Rightarrow 2y = -\frac{1}{2}x^2 + \alpha$ （ $\alpha = 0$ ）

$\Rightarrow 2y = -\frac{1}{2}x^2$ …singular solution

---

⑦ $y = xp - e^p$ ， $p = \frac{dy}{dx}$ ，求通解及奇異解

$\Rightarrow y' = y' + xy'' - e^{y'}y''$

$\Rightarrow y''(x - e^{y'}) = 0$

$\Rightarrow$ (1) $y'' = 0 \Rightarrow y' = c \Rightarrow y = cx - e^c$

(2) $x - e^{y'} = 0 \Rightarrow e^{y'} = x \Rightarrow y = xp - e^p = xy' - x$

$\Rightarrow$ 取 ln $\Rightarrow$ $y' = \ln x \Rightarrow y = x\ln x - x + \alpha$

$\Rightarrow c = \ln x$ ， $y = cx - x + \alpha$

$\Rightarrow cx - e^c = cx - x + \alpha$

$\Rightarrow \alpha = 0$

7.Special types of second-order equation

① $t\frac{d^2x}{dt^2} = 2[(\frac{dx}{dt})^2 - \frac{dx}{dt}] \Rightarrow$ Let $y = \frac{dx}{dt}$

$\Rightarrow t\frac{dy}{dt} = 2[y^2 - y]$

$\Rightarrow x = \frac{1}{c_1}\tan^{-1} c_1 t + c_2$

---

② $4xy'^2 + 2xy' - y = 0 \Rightarrow 8xy'y'' + 4y'^2 + 2y' + 2xy'' - y' = 0$

$\Rightarrow 2xy''(4y' + 1) + y'(4y' + 1) = 0$

$\Rightarrow (4y' + 1)(2xy'' + y') = 0$

$$\Rightarrow \ y=-\frac{x}{4} \text{，} y=4c^2+2c\sqrt{x}$$

8.Orthogonal trajectories

① a family of parabolas

$$y=cx^2 \ \Rightarrow \ c=\frac{y}{x^2}$$

$$y'=2cx=2(\frac{y}{x^2})x$$

$$y'=\frac{2y}{x}$$

Orthogonal trajectories

$$y'=-\frac{x}{2y}$$ （∵兩垂直直線斜率相乘為 -1）

積分 $\Rightarrow \ \frac{x^2}{2}+y^2=c^*$ ………ellipse（橢圓）

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② $x^2+(y-c)^2=c^2$，求正交曲線

微分 $\Rightarrow \ 2x+2(y-c)y'=0$

$$\Rightarrow \ x^2+y^2-2cy+c^2=c^2$$

$$\Rightarrow \ c=\frac{1}{2y}(x^2+y^2)$$

$$\Rightarrow \ y'=\frac{2xy}{x^2-y^2}$$

正交 $\Rightarrow \ y'=-\frac{x^2-y^2}{2xy}$

$$\Rightarrow \ 2xyy'=-x^2+y^2$$

積分 $\Rightarrow \ (x-c')^2+y^2=c'^2$
