High Order Linear Differential Equation

1.Linear Differential equation

一般形式

$$y^{(n)}+P_{n-1}(x)y^{(n-1)}+\dots+P_1(x)y'+P_0(x)y=r(x)$$

$$Ly=r$$

$$L\equiv D^n+P_{n-1}(x)D^{n-1}+\dots+P_1(x)D+P_0(x)$$

$$D\equiv\frac{d}{dx}$$

$\Rightarrow$ $r(x)=0$ …homogeneous

$r(x)\neq 0$ …nonhomogeneous

L 滿足 $L(cy)=cL(y)$

$$L(y_1+y_2)=L(y_1)+L(y_2)$$

稱 L 爲 linear operator

y 的解分成 $y=y_h+y_p$（homogeneous + particular）

$$y_h=c_1y_1+c_2y_2+c_3y_3+\dots c_ny_n$$

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① $y''-3y'+2y=0\ \Rightarrow\ y_1=e^x$ 成立

$y_2=e^{2x}$ 成立

$\Rightarrow\ y=c_1e^x+c_1e^{2x}$ 亦成立

Linear Dependence

n 個函數 y_1、y_2、……、y_n 爲線性相關

$c_1y_1+c_2y_2+c_3y_3+\dots c_ny_n=0$ 時，存在 c_1、c_2、…、c_n 不全爲 0，否則稱爲線性獨立（linear independence）

若 y_1、y_2、……、y_n 爲線性相關，若且唯若至少有一函數可以表成其他函數得線性組合（linear combination）說明若 $c_1y_1+c_2y_2+c_3y_3+\dots c_ny_n=0$，若 $c_k\neq 0$，則

$$y_k=-\frac{c_1}{c_k}y_1-\dots-\frac{c_{k-1}}{c_k}y_k-\dots-\frac{c_n}{c_k}y_n$$

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② $y_1=e^x$，$y_2=e^{2x}$ 對任意 x 有 y_1 和 y_2 線性獨立？

PS：$\begin{cases}c_1y_1+c_2y_2=0\\ c_1y_1'+c_2y_2'=0\end{cases}\Rightarrow c_1$、$c_2$ 有非零解，則 $\omega=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$

If $\omega=0$ 則獨立

$$\omega=\begin{vmatrix}e^x & e^{2x}\\ e^x & 2e^{2x}\end{vmatrix}=e^{2x}\neq 0$$

c_1、c_2均為零 $\Rightarrow$ y_1和y_2獨立。

Wronskian

有y_1、y_2、……、y_n函數，定義 wronskian

$$\omega = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

若對任一x，$\omega \neq 0$，則y_1、y_2、……、y_n線性獨立

若y_1、y_2、……、y_n線性相關，則$\omega = 0$。（反之不一定成立）

③ $y_1 = e^{\lambda_1 x}$，$y_2 = e^{\lambda_2 x}$，$y_3 = e^{\lambda_3 x}$，…，$y_n = e^{\lambda_n x}$

$$\omega = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} = e^{(\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

$$\left(D_n \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} \right)$$

$$\Rightarrow \quad D_n = (-1)^{\frac{n(n-1)}{2}} \prod_{j>i} (\lambda_i - \lambda_j)$$

所有λ_i都不相同 $\Rightarrow$ y_1、y_2、……、y_n為線性獨立

④ $y_1 = \cos x$，$y_2 = \sin x$

$$\Rightarrow \quad \omega = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ y_1和y_2線性獨立

⑤ $y_1 = x^2$，$y_2 = x|x|$是否為線性獨立？

$$\Rightarrow \quad x \geq 0，\omega = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$$

$$\Rightarrow \quad x < 0，\omega = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$$

$$\Rightarrow \quad c_1 x^2 + c_2 x|x| = 0$$

$$\Rightarrow \quad \begin{cases} x = 1 & c_1 + c_2 = 0 \\ x = -1 & c_1 - c_2 = 0 \end{cases} \quad \Rightarrow \quad c_1 = c_2 = 0$$

$\Rightarrow$ y_1和y_2線性相關

2.Homogeneous Equation with Constant Coefficients

一般形式

$y^{(n)}+a_{n-1}y^{(n-1)}+......+a_1y'+a_0y=0$

猜 $y=e^{\lambda x}$ 代入上式

$\Rightarrow \lambda^n+a_{n-1}\lambda^{n-1}+......+a_1\lambda+a_0=0$

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① $y''-3y'+2y=0$（λ的根爲相異實數）

$\Rightarrow$ Let $y=e^{\lambda x}$

$\Rightarrow \lambda^2e^{\lambda x}-3\lambda e^{\lambda x}+2e^{\lambda x}=0$

$\Rightarrow \lambda^2-3\lambda+2=0$

$\Rightarrow (\lambda-2)(\lambda-1)=0 \Rightarrow \lambda=1,2$

$\Rightarrow y=c_1e^x+c_2e^{2x}$

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② $y''+y=0$（λ的根爲共軛複根）

$\Rightarrow y=e^{\lambda x}$

$\Rightarrow \lambda^2+1=0 \Rightarrow \lambda=\pm i$

$\Rightarrow y=c_1e^{ix}+c_2e^{-ix}$ （$e^{ix}=\cos x+i\sin x$）

$=c_1(\cos x+i\sin x)+c_2(\cos x-i\sin x)$

$=a_1\cos x+a_2\sin x$ （$a_1=c_1+c_2$，$a_2=ic_1-ic_2$）

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③ $y'''-5y''+9y'-5y=0 \Rightarrow$ Let $y=e^{\lambda x}$

$\Rightarrow \lambda^3-5\lambda^2+9\lambda-5=0$

$\Rightarrow (\lambda-1)(\lambda^2-4\lambda+5)=0$

$\Rightarrow \lambda=2+i,2-i,1$

$\Rightarrow y=c_1e^x+c_2e^{(2+i)x}+c_3e^{(2-i)x}=c_1e^x+c_2e^{2x}\cos x+c_3e^{2x}\sin x$

重根（multiple roots）

當係數微分方程中，λ有重根，$\lambda=\lambda_1=\lambda_2=......$

則解 $y=(c_1+c_2x+c_3x^2+......)e^{\lambda x}$

說明：以二重根爲例，當$\lambda=\lambda_1=\lambda_2$時

$$y=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}=c_1'e^{\lambda_1x}+c_2'\left(\frac{e^{\lambda_1x}-e^{\lambda_2x}}{\lambda_1-\lambda_2}\right)$$

$$c_1=c_1'+\frac{c_2'}{\lambda_1-\lambda_2}$$

$$c_2=-\frac{c_2'}{\lambda_1-\lambda_2}$$

$$\lambda_1 \to \lambda_2 = \lambda \quad \Rightarrow \quad y = c_1' e^{\lambda x} + c_2' x e^{\lambda x}$$

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④ $y''' + 4'' - 3y' - 18y = 0 \;\Rightarrow \text{Let} \;\; y = e^{\lambda x}$

$\Rightarrow \;\; \lambda^3 + 4\lambda^2 - 3\lambda - 18 = 0$

$\Rightarrow \;\; (\lambda + 3)^2 (\lambda - 2) = 0 \;\;\Rightarrow\;\; \lambda = -3, -3, 2$

$\Rightarrow \;\; y = c_1 e^{-3x} + c_2 x e^{-3x} + c_3 e^{2x}$

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⑤ $(D^4 + 8D^2 + 16) y = 0 \;\;\Rightarrow\;\; \lambda^4 + 8\lambda^2 + 16 = 0$

$\Rightarrow \;\; (\lambda^2 + 4)^2 = 0 \;\;\Rightarrow\;\; \lambda = 2i, 2i, -2i, -2i$

$\Rightarrow \;\; y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

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⑥ $y''' - 3y'' + 3y' - y = 0$

$\Rightarrow \;\; \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$\Rightarrow \;\; (\lambda - 1)^3 = 0 \;\;\Rightarrow\;\; \lambda = 1,1,1$

$\Rightarrow \;\; y = (c_1 + c_2 x + c_3 x^2) e^x$

if $\;y(0) = y'(0) = 0$ ， $y''(0) = 2$

$y(0) = 0 \;\;\Rightarrow\;\; c_1 = 0$

$y'(x) = c_2 e^x + c_2 x e^x + c_3 (2xe^x + x^2 e^x)$

$\Rightarrow \;\; y'(0) = 0 \;\;\Rightarrow\;\; c_2 = 0$

$y''(x) = c_3 (2e^x + 2xe^x + 2xe^x + x^2 e^x)$

$\Rightarrow \;\; y''(0) = 2 = 2c_3 \;\;\Rightarrow\;\; c_3 = 1$

$y = x^2 e^x$

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⑦ $y^{(7)} + 18 y^{(5)} + 81 y''' = 0$

$\Rightarrow \;\; \lambda^7 + 18\lambda^5 + 81\lambda^3 = 0$

$\Rightarrow \;\; \lambda^3 (\lambda + 3i)^2 (\lambda - 3i)^2 = 0 \;\;\Rightarrow\;\; \lambda = 0,0,0,3i,3i,-3i,-3i$

$\Rightarrow \;\; y = c_1 + c_2 x + c_3 x^2 + c_4 \cos 3x + c_5 \sin 3x + c_6 x \cos 3x + c_7 x \sin 3x$

3.Cauchy-Euler Equation

$$(x^n D^n + b_{n-1} x^{n-1} D^{n-1} + \ldots\ldots + b_1 x D + b_0) y = 0$$

$$y \approx x^{\alpha}$$

① $2x^2 y'' - 5xy' + 3y = 0 \;\Rightarrow \text{Let} \;\; y = x^{\alpha} \;\;\Rightarrow\;\; y'' = \alpha(\alpha - 1) x^{\alpha - 2}$

$\Rightarrow \;\; 2\alpha(\alpha - 1) - 5\alpha + 3 = 0$

$\Rightarrow \;\; 2\alpha^2 - 7\alpha + 3 = 0 \;\;\Rightarrow\;\; \alpha = \frac{1}{2}, 3$

$\Rightarrow \;\; y = c_1 x^{\frac{1}{2}} + c_2 x^3$

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② $x^2 y'' - xy' + 5y = 0$

$$\Rightarrow \text{Let} \quad y = x^{\alpha}$$

$$\Rightarrow \quad \alpha(\alpha-1)-\alpha+5=0 \quad \Rightarrow \quad \alpha^2-2\alpha+5=0 \quad \Rightarrow \quad \alpha = 1 \pm 2i$$

$$y = a_1 x^{1+2i} + a_2 x^{1-2i}$$

$$x^{2i} = e^{\ln x^{2i}} = e^{2i\ln x} = \cos(2\ln x) + i\sin(2\ln x)$$

$$\Rightarrow \quad y = a_1 x[\cos(2\ln x) + i\sin(2\ln x)] + a_2 x[\cos(2\ln x) - i\sin(2\ln x)]$$

$$= c_1 x cox(2\ln x) + c_2 x \sin(2\ln x)$$

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③ $x^2 y'' - 2xy' + 9y = 0 \;\Rightarrow \text{Let} \quad y = x^{\alpha}$

$$\Rightarrow \quad \alpha(\alpha-1)-2\alpha+9=0 \quad \Rightarrow \quad \alpha^2-3\alpha+9=0 \quad \Rightarrow \quad \alpha = \frac{3}{2} \pm \frac{3}{2}\sqrt{3}i$$

$$\Rightarrow \quad y = c_1 x^{\frac{3}{2}} \cos\left(\frac{3\sqrt{3}}{2}\ln x\right) + c_2 x^{\frac{3}{2}} \sin\left(\frac{3\sqrt{3}}{2}\ln x\right)$$

重根 $\lambda = \lambda_1 = \lambda_2$

$$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$$

$$= c_1' x^{\lambda_1} + c_2' \frac{x^{\lambda_1} - x^{\lambda_1}}{\lambda_1 - \lambda_2}$$

$$\xrightarrow[\text{對}\lambda_1\text{微分}]{\lambda_1 \to \lambda_2 \equiv \lambda} \quad \Rightarrow \quad c_1' x^{\lambda} + x_2' x^{\lambda}(\ln x)$$

$$\left(\frac{d}{d\lambda_1} x^{\lambda_1} = \frac{d}{d\lambda_1} e^{\lambda_1 \ln x} = (\ln x) x^{\lambda_1} \right)$$

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④ $x^3 y''' - x^2 y'' + xy' = 0 \;\Rightarrow \text{Let} \quad y = x^{\alpha}$

$$\Rightarrow \quad \alpha(\alpha-1)(\alpha-2) - \alpha(\alpha-1) + \alpha = 0$$

$$\Rightarrow \quad \alpha[\alpha^2 - 3\alpha + 2 - \alpha + 1 + 1] = 0$$

$$\Rightarrow \quad \alpha(\alpha-2)^2 = 0 \quad \Rightarrow \quad \alpha = 0,2,2$$

$$\Rightarrow \quad y(x) = c_1 + c_2 x^2 + c_3 x^2 (\ln x)$$

4.Nonhomogeneous Equation

$Ly(x) = r(x)$ $\quad r(x) \neq 0$ $\quad L$：係數，cauchy-eular 型

$Ly_h(x) = 0$

$y = y_h + y_p$ $\quad p$：particular

① $y'' - 4y' + 4y = 9e^{-x} \;\Rightarrow\; \lambda^2 - 4\lambda + 4 = 0 \;\Rightarrow\; \lambda = 2,2$

$$\Rightarrow \quad y_h = (c_1 + c_2 x)e^{2x}$$

$$\Rightarrow \quad y_p = ae^{-x} \quad \Rightarrow \quad a(-1)^2 - 4a(-1) + 4a = 9 \quad \Rightarrow \quad a = 1$$

$$\Rightarrow \quad y = (c_1 + c_2 x)e^{2x} + e^{-x}$$

② $y''-y=-5\sin 2x \;\Rightarrow\; \lambda^2-1=0 \quad\Rightarrow\; \lambda=1,-1$

$\Rightarrow\; y_h=c_1e^x+c_2e^{-1}$

$\Rightarrow\; y_p=a\sin 2x \;\Rightarrow\; y_p'=2a\cos 2x \;\Rightarrow\; y_p''=-4a\sin 2x$

$\Rightarrow\; -4a-a=-5 \;\Rightarrow\; a=1$

$\Rightarrow\; y=c_1e^x+c_2e^{-1}+\sin 2x$

Method of Ondertermined Coefficient

$Ly=r(x)$

$y=y_h+y_p$

原則上 $r(x)=c$ $\qquad y_p=c_1$

$r(x)=x^n$ $\qquad y_p=c_nx^n+c_{n-1}x^{n-1}+\ldots\ldots+c_1x+c_0$

$r(x)=e^{ax}$ $\qquad y_p=c_1e^{ax}$

$r(x)=\sin bx$ $\qquad y_p=c_1\sin bx+c_2\cos bx$

$r(x)=\cos bx$ $\qquad y_p=c_1\sin bx+c_2\cos bx$

③ $y''-2y'+y=6\sin x$

$\Rightarrow\; y_h=(c_1+c_2x)e^x$

$\Rightarrow\; y_p=a\sin x+b\cos x$

$\Rightarrow\; y_p'=a\cos x-b\sin x$

$\Rightarrow\; y_p''=-a\sin x-b\cos x$

$\Rightarrow\; (-a\sin x-b\cos x)-2(a\cos x-b\sin x)+(a\sin x+b\cos x)=6\sin x$

$\Rightarrow\; a=0$ ， $b=3$

$\Rightarrow\; y=(c_1+c_2x)e^x+3\cos x$

④ $y'''+3y''+4y'+2y=170\cos 3x$

$\Rightarrow\; \lambda^3+3\lambda^2+4\lambda+2=0$

$\Rightarrow\; \lambda(\lambda+1)(\lambda+2)+2(\lambda+1)=0$

$\Rightarrow\; (\lambda+1)(\lambda^2+2\lambda+2)=0 \;\Rightarrow\; \lambda=-1,-1\pm 2i$

$\Rightarrow\; y_h=c_1e^{-x}+c_2e^{-x}\cos 2x+c_2e^{-x}\sin 2x$

$\Rightarrow\; y_p=a\cos 3x+b\sin 3x$ $\qquad$ 代入 $y'''+3y''+4y'+2y=170e^{i3x}$

$\Rightarrow\; [(3i)^3+3(3i)^2+4(3i)+2]A=170$

$$\Rightarrow\; A=\frac{170}{-25-15i}=\frac{34}{-5-3i}\cdot\frac{5-3i}{5-3i}=-(5-3i)$$

$\Rightarrow\; y_0=Ae^{i3x}=-(5-3i)e^{i3x}=(-5+3i)(\cos 3x+i\sin 3x)$

$\Rightarrow\; y_p=\operatorname{Re} y_0=-5\cos 3x-3\sin 3x$

未定係數法修正

若 $Ly = Ax^j e^{cx}$

(1) c 和左式 Ly 所得根不同者

$$y_p = e^{cx}(A_j x^j + \ldots\ldots + A_1 x + A_0)$$

(2) c 和左式 Ly 所得根相同者（有 m 次相同）

$$y_p = x^m(A_j x^j + \ldots\ldots + A_1 x + A_0)e^{cx}$$

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⑤ $y'' - 4y' + 4y = 12xe^{2x}$

$\Rightarrow$ Let $y_h = e^{\lambda x}$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2,2$

$\Rightarrow y_h = c_1 e^{2x} + c_2 xe^{2x}$

$\Rightarrow y_p = x^2(A_1 x + A_0)e^{2x}$

$\Rightarrow y_p' = (3A_1 x^2 + 2A_0 X)e^{2x} + (A_1 x^3 + A_0 x^2)2e^{2x}$

$= [2A_1 x^3 + (3A_1 + 2A_0)x^2 + 2A_0 x]e^{2x}$

$\Rightarrow y_p''' = [6A_1 x^2 + (6A_1 + 4A_0)x + 2A_0]e^{2x} + [2A_1 x^3 + (3A_1 + 2A_0)x^2 + 2A_0 x]2e^{2x}$

$\Rightarrow x^3 e^{2x}$ 的係數： $4A_1 - 4(2A_1) + 4A_1 = 0$

$x^2 e^{2x}$ 的係數： $2(3A_1 + 2A_0) + 6A_1 - 4(3A_1 + 2A_0) + 4A_0 = 0$

xe^{2x} 的係數： $4A_0 + (6A_1 + 4A_0) - 4 \cdot 2A_0 = 12 \Rightarrow A_1 = 2$

e^{2x} 的係數： $2A_0 = 0 \Rightarrow A_0 = 0$

$\Rightarrow y = (c_1 + c_2 x)e^{2x} + 2x^3 e^{2x}$

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⑥ $y'' + 2y' + y = xe^{-x}$，$y(0) = 0$，$y'(0) = 1$

$\Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1,1$

$\Rightarrow y_h = (c_1 + c_2 x)e^{2x}$

$\Rightarrow y_p = x^2(A_1 x + A_0)e^{-x}$

$\Rightarrow y = c_1 e^{-x} + c_2 xe^{2x} + \dfrac{x^3}{6}e^{-x}$

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⑦ $y''' - y'' = 2\sin x - 5e^x \Rightarrow \lambda^3 - \lambda^2 = 0 \Rightarrow \lambda = 0,0,1$

$\Rightarrow y_h = c_1 + c_2 x + c_3 e^x$

$\Rightarrow y_p = a\sin x + b\cos x + Ae^x x$

$\Rightarrow a = 1$，$b = 1$，$A = -5$

$\Rightarrow y = c_1 + c_2 x + c_3 e^x + \sin x + \cos x - 5xe^x$

Variation of Parameters（參數變更法）

① $y''-3y'+2y=-\dfrac{e^{2x}}{e^{x}+1}$

$\Rightarrow$ Let $y_h''-3y_h'+2y_h=0$

$\Rightarrow$ Let $y_h=e^{\lambda x}$

$\Rightarrow\ \alpha^2-3\alpha+2=0\ \Rightarrow\ \alpha=1,2$

$\Rightarrow\ y_h=c_1e^{x}+c_2e^{2x}$

設 $y_p=c_1(x)e^{x}+c_2(x)e^{2x}$

$y_p'=c_1e^{x}+2c_2e^{2x}+c_1'e^{x}+c_2'e^{2x}$

$y_p''=c_1e^{x}+4c_2e^{2x}+c_1'e^{x}+2c_2'e^{2x}$

$$\Rightarrow\ \begin{cases} c_1'e^{x}+c_2'e^{2x}=0 \\ c_1'e^{x}+2c_2'e^{2x}=-\dfrac{e^{2x}}{e^{x}+1} \end{cases}$$

$\Rightarrow\ c_1'=\dfrac{e^{x}}{e^{x}+1}$ ， $c_2'=-\dfrac{1}{e^{x}+1}$

$\Rightarrow\ c_1=\ln(e^{x}+1)$ ， $c_2=\ln(1+e^{-x})$

$\Rightarrow\ y=c_1e^{x}+c_2e^{2x}+e^{x}\ln(e^{x}+1)+e^{2x}\ln(1+e^{-x})$

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② $y''+2y'+y=\dfrac{1}{x}e^{-x}$

$\Rightarrow\ \lambda^2+2\lambda+1=0\ \Rightarrow\ \lambda=-1,-1$

$\Rightarrow\ y_h=c_1e^{-x}+c_2xe^{-x}$

$\Rightarrow\ y_p=c_1(x)e^{-x}+c_2(x)xe^{-x}$

$y_p'=c_1(-e^{-x})+c_2(e^{-x}-xe^{-x})+c_1'e^{-x}+c_2'xe^{-x}$

$y_p''=c_1e^{-x}+c_2(-e^{-x})-c_2e^{-x}+c_2xe^{-x}-c_1'e^{-x}+c_2'(e^{-x}-xe^{-x})$

$\Rightarrow\ c_1'e^{-x}+c_2'xe^{-x}=0$

$-c_1'e^{-x}+c_2'(e^{-x}-xe^{-x})=\dfrac{1}{x}e^{-x}$

$\Rightarrow\ c_2'e^{-x}=\dfrac{1}{x}e^{-x}\ \Rightarrow\ c_2'=\dfrac{1}{x}\ \Rightarrow\ c_2=\ln x$

$c_1'=-1\ \Rightarrow\ c_1=-x$

$\Rightarrow\ y=c_1e^{-x}+c_2xe^{-x}+x(\ln x)e^{-x}$

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③ $x^2y''-xy'+y=x\ln x$

$\Rightarrow$ Let $y_h=e^{\alpha}$

$\Rightarrow\ \alpha(\alpha-1)-\alpha+1=0\ \Rightarrow\ \alpha=1,1$

$\Rightarrow\ y_h=c_1x+c_2x\ln x$

$\Rightarrow\ y_p=c_1(x)x+c_2(x)x\ln x$

$\Rightarrow\ y_p'=c_1+c_2(\ln x+1)+c_1'x+c_2'x\ln x$

$$\Rightarrow\ y_p'' = c_1' + c_2\left(\frac{1}{x}\right) + c_2'(\ln x + 1)$$

$$\Rightarrow\ \begin{cases} c_1'x + c_2'x\ln x = 0 \\ x^2[c_1' + c_2'(\ln x + 1)] = x\ln x \end{cases}$$

$$\Rightarrow\ c_2' = \frac{\ln x}{x} \text{ ， } c_1' = -\frac{(\ln x)^2}{x}$$

$$\Rightarrow\ c_2 = \frac{1}{2}(\ln x)^2 \text{ ， } c_1 = -\frac{1}{3}\frac{(\ln x)^3}{x}$$

$$\Rightarrow\ y_p = [-\frac{1}{3}(\ln x)^3] + [\frac{1}{2}(\ln x)^2](x\ln x) = \frac{1}{6}x(\ln x)^3$$

$$\Rightarrow\ y = c_1x + c_2x\ln x + \frac{1}{6}x(\ln x)^3$$

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④ $y'' + y = \sec x$

$$\Rightarrow\ \lambda^2 + 1 = 0 \ \Rightarrow\ \lambda = \pm i$$

$$\Rightarrow\ y_h = c_1\cos x + c_2\sin x$$

$$\Rightarrow\ y_p = c_1(x)\cos x + c_2(x)\sin x$$

$\Rightarrow$ …

$\Rightarrow$ ……

$$\Rightarrow\ \begin{cases} c_1'\cos x + c_2'\sin x = 0 \\ c_1'(-\sin x) + c_2'\cos x = \sec x \end{cases}$$

$$\Rightarrow\ c_1' = -\frac{\sin x}{\cos x} \text{ ， } c_2' = 1$$

$$\Rightarrow\ c_1 = \ln\cos x \text{ ， } c_2 = x$$

$$\Rightarrow\ y = c_1\cos x + c_2\sin x + x\sin x + \cos x(\ln\cos x)$$

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⑤ $y'' + y = \sec x$

$$\Rightarrow\ \lambda^2 + 1 = 0 \ \Rightarrow\ \lambda = \pm i$$

$$\Rightarrow\ y_h = c_1\cos x + c_2\sin x$$

$$\Rightarrow\ y_p = c_1(x)\cos x + c_2(x)\sin x$$

$$\Rightarrow\ \begin{cases} c_1'\cos x + c_2'\sin x = 0 \\ c_1'(-\sin x) + c_2'\cos x = \csc x \end{cases}$$

$$\Rightarrow\ c_1' = -1 \text{ ， } c_2' = \frac{\cos x}{\sin x}$$

$$\Rightarrow\ c_1 = -x \text{ ， } c_2 = \ln(\sin x)$$

$$\Rightarrow\ y = c_1\cos x + c_2\sin x - x\cos x + (\ln\sin x)\sin x$$

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⑥ $xy'' - 3xy' - 5y = 6x^5$

$\Rightarrow$ Let $y_h = x^{\alpha}$

$\Rightarrow \alpha(\alpha-1)-3\alpha-5=0 \Rightarrow \alpha=5,-1$

$\Rightarrow y_h = c_1x^{-1}+c_2x^5$

$\Rightarrow y_p = c_1(x)x^{-1}+c_2(x)x^5$

$$\Rightarrow \begin{cases} c_1'x^{-1}+c_2'x^5=0 \\ x^2[c_1'(-\frac{1}{x^2})+c_2'(5x^4)]=6x^5 \end{cases}$$

$\Rightarrow c_2' = \frac{1}{x} \Rightarrow c_2 = \ln x$

$\Rightarrow c_1' = -x^5 \Rightarrow c_1 = -\frac{1}{6}x^6$（$c_1(x)x^{-1}$亦是$x^5$，所以$-\frac{1}{6}x^5$可併入$c_2(x)x^5$裡）

$\Rightarrow y = c_1x^{-1}+c_2x^5+(-\frac{1}{6}+\ln x)x^5$

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⑦ $y''-\frac{2}{x}y'+\frac{2}{x^2}y = x\ln x$

$\Rightarrow$ Let $y_h = x^{\alpha}$

$\Rightarrow \alpha(\alpha-1)-2\alpha+2=0 \Rightarrow \alpha=1,2$

$\Rightarrow y_h = c_1x+c_2x^2$

$\Rightarrow y_p = c_1(x)x+c_2(x)x^2$

$$\Rightarrow \begin{cases} c_1'x+c_2'x^2=0 \\ c_1'+2c_2'x = x\ln x \end{cases}$$

$\Rightarrow c_2' = \ln x \Rightarrow c_2 = x\ln x - x$

$\Rightarrow c_1' = -x\ln x \Rightarrow c_1 = -\frac{1}{2}x^2\ln x+\frac{1}{4}x^2$

$\Rightarrow y = c_1x+c_2x^2+x(-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2)+x^2(x\ln x-x)$

$= c_1x+c_2x^2+\frac{1}{2}x^3\ln x-\frac{3}{4}x^3$

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⑧ $x^2(x+1)y''-2xy'+2y=0$　　已知$y=x$爲一解

$\Rightarrow$ Let $y = cx$ ， $y' = c'x + c$ ， $y'' = c''x + 2c'$

$\Rightarrow x^2(x+1)(c''x + 2c') - 2x(c'x + c) + 2cx = 0$

$\Rightarrow x^4c'' + 2x^3c' + x^3c'' + x^2 2c' - 2c'x^2 - 2cx + 2cx = 0$

$\Rightarrow (x+1)c'' + 2c' = 0$

$\Rightarrow$ Let $c' = u \Rightarrow (x+1)u' + 2u = 0$

$$\Rightarrow \frac{du}{dx} = -\frac{2u}{x+1}$$

$$\Rightarrow \frac{1}{u}du = -2\frac{1}{x+1}dx$$

$\Rightarrow \ln u = -2\ln(x+1) + d$

$\Rightarrow u = d_1(x+1)^{-2} = c'$ （ $d_1 = e^d$ ）

$$\Rightarrow c = -d_1\frac{1}{x+1} + e$$

$\Rightarrow y = cx$

$$\Rightarrow y = x(-d_1\frac{1}{x+1} + e) = c_1\frac{x}{x+1} + c_2x$$ （ $-d_1 = c_1$ ， $e = c_2$ ）