級數解法

1.Power Series Solution

$$P(x)y''+Q(x)y'+R(x)y=0$$

ordinary point $y=\sum_{n=0}^{\infty}a_nx^n$	$\Rightarrow$ recurrence relation $\Rightarrow a_n$
Regular singular point $y=x^s\sum_{n=0}^{\infty}a_nx^n$	$\Rightarrow$ indicial（指標 eq.）$\Rightarrow$ recurrence relation $\Rightarrow a_n$

$$y''+\frac{Q(x)}{P(x)}y'+\frac{R(x)}{P(x)}y=0$$

P、Q、R 代入 x 不會發散

$P(x_0)\neq 0$，$x=x_0$，ordinary point

$P(x_0)=0$，$x=x_0$，singular point（奇異點）

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① $y''+y=0$（$x=0$是 ordinary point）

$$\Rightarrow y=\sum_{n=0}^{\infty}a_nx^n \Rightarrow y'=\sum_{n=0}^{\infty}na_nx^{n-1} \Rightarrow y''=\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\Rightarrow \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+a_n]x^n=0 \qquad ((n+2)(n+1)a_{n+2}+a_n=0)$$

$$\Rightarrow a_{n+2}=-\frac{1}{(n+2)(n+1)}a_n \text{ ……recurrence relation}$$

$\Rightarrow n=0,1,2,3,……$

$n=0$，$a_2=-\frac{1}{2\cdot 1}a_0$ $\qquad$ $n=1$，$a_3=-\frac{1}{3!}a_1$

$n=2$，$a_4=-\frac{1}{4\cdot 3}a_2=(-\frac{1}{4\cdot 3})(-\frac{1}{2\cdot 1})a_0$

$n=4$，$a_6=-\frac{1}{6!}a_0$

$$a_{2m}=-\frac{(-1)^m}{(2m)!}a_0 \qquad a_{2m+1}=\frac{(-1)^m}{(2m)!}a_1$$

紋的筆記

② $y'' - 2xy' - y = 0$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - 2n a_n - a_n] x^n = 0$$

$$2a_2 - a_0 = 0$$

$$(n+2)(n+1) a_{n+2} - 2n a_n - a_n = 0$$

$$\Rightarrow a_2 = \frac{1}{2} a_0 \text{，} a_{n+2} = \frac{(2n+1)}{(n+2)(n+1)} a_n \quad n \ge 0$$

$$\Rightarrow a_4 = \frac{5}{4\cdot 3} a_2 = \frac{5}{4\cdot 3\cdot 2} a_0$$

$$a_6 = \frac{9}{6\cdot 5} a_4 = \frac{9\cdot 5}{6!} a_0$$

$$\cdots\cdots$$

$$a_{2m} = \frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4m-3)}{(2m)!} a_0 \text{（} m \ge 1 \text{）}$$

$$\Rightarrow a_3 = \frac{3}{3\cdot 2} a_1$$

$$a_5 = \frac{7}{5\cdot 4} a_3 = \frac{3\cdot 7}{5!} a_1$$

$$\cdots\cdots$$

$$a_{(2m+1)} = \frac{3\cdot 7\cdot \ldots \cdot (4m-1)}{(2m+1)!} a_1 \text{（} m \ge 1 \text{）}$$

$$\Rightarrow y = a_0 + a_2 x^2 + a_4 x^4 + \ldots\ldots + a_1 x + a_3 x^3 + a_5 x^5 + \ldots\ldots$$

$$\Rightarrow y = a_0 [1 + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4m-3)}{(2m)!} x^{2m}] + a_1 x [1 + \sum_{n=1}^{\infty} \frac{3\cdot 7\cdot \ldots \cdot (4m-1)}{(2m+1)!} x^{2m-1}]$$

$$a_0 + \frac{1}{2} a_0 + \frac{5}{4!} a_0 + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4m-3)}{(2m)!} x^{2m} a_0$$

$$= a_0 [1 + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4m-3)}{(2m)!} x^{2m}]$$

2.Regular Singular Points

$$P(x) y'' + Q(x) y' + R(x) y = f(x)$$

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = \frac{f(x)}{P(x)}$$

$P(x)$、$Q(x)$、$R(x)$、x 任何點都不會發散

$P(x = x_0) = 0$ （x_0 ：singular point）

$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)}$ 存在，且 $\lim_{x \to x_0} (x - x_0)^2 \frac{Q(x)}{P(x)}$ 存在

則 $x = x_0$ 是 regular singular point

紋的筆記

$$y''+\frac{a(x)}{(x-x_0)}y'+\frac{b(x)}{(x-x_0)^2}y=c(x)$$

$a(x)$ 、 $b(x)$ 任何 x 代入

Frobenius method $y=x^r\sum_{n=0}^{\infty}a_nx^n$

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① $2x^2y''+3xy'-(1+x)y=0$ ， $x=0$ is a regular singular point

Let $y=x^r\sum_{n=0}^{\infty}a_nx^n \Rightarrow y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} \Rightarrow y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+3\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}=0$$

$$\Rightarrow [2r(r-1)+3r-1]a_0x^r+\sum_{n=1}^{\infty}\{[2(n+r)(n+r-1)+3(n+r)-1]a_n-a_{n-1}\}x^{n+r}=0$$

$\Rightarrow 2r(r-1)+3r-1=(2r-1)(r+1)=0$ （指標 eq.）

$\Rightarrow r=r_1=\frac{1}{2}$ ， $r=r_2=-1$

$$\Rightarrow r=r_1=\frac{1}{2} \Rightarrow a_n=\frac{1}{n(2n+3)}a_{n-1} \Rightarrow a_1=\frac{1}{1\cdot5}a_0$$

$$\Rightarrow a_2=\frac{1}{2\cdot7}a_1=\frac{1}{1\cdot2\cdot5\cdot7}a_0$$

$$\Rightarrow a_3=\frac{1}{3\cdot9}a_2$$

$\Rightarrow \ldots\ldots$

$$\Rightarrow a_n=\frac{1}{n!5\cdot7\cdot\ldots\cdot(2n+3)}a_0$$

$$\Rightarrow y_1=x^{\frac{1}{2}}[1+\sum_{n=1}^{\infty}\frac{x^n}{n!5\cdot7\cdot\ldots\cdot(2n+3)}]$$

$$\Rightarrow r=r_2=-1 \Rightarrow a_n=\frac{1}{n(2n-3)}a_{n-1}=\frac{1}{n!(-1)\cdot1\cdot3\cdot\ldots\cdot(2n-3)}a_0$$

$$\Rightarrow y_2=x^{-1}[1-x-\sum_{n=2}^{\infty}\frac{x^n}{n!1\cdot3\cdot\ldots\cdot(2n-3)}]$$

$\Rightarrow y=c_1y_1+c_2y_2$

Equal exponents

$Ly(x,r)=0$ ， $r_1=r_2$

$y^*=x^r\sum_{n=0}^{\infty}a_n(r)x^n$ （ y^* 中的 r 爲一變數）

$$y_1=y^*(x,r)\Big|_{r=r_1}=x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n$$

紋的筆記-紋的筆記

$$y_2 = \frac{\partial y^*}{\partial r}\Big|_{r=r_1} = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n \ln x + \sum_{n=0}^{\infty} a_n'(r_1)x^n$$

$$\left(\frac{d}{dr}x^r = \frac{d}{dr}e^{r\ln x} = (\ln x)x^r \right)$$

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② $xy'' + y' - y = 0$ ， $x = 0$ is a regular singular point

Let $y = x^r\sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \Rightarrow y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow [r(r-1)+r]a_0 x^{r-1} + \sum_{n=1}^{\infty}\{[(n+r)(m-n+r-1)+(n+r)]a_n - a_{n-1}\}x^{n+r-1} = 0$$

$$\Rightarrow r^2 = 0 \Rightarrow r = 0,0$$

$$\Rightarrow a_n = \frac{1}{(n+r)^2}a_{n-1} = \frac{1}{(n+r)^2......(1+r)^2}a_0$$

$$\Rightarrow y^* = a_0 x^r[1 + \sum_{n=1}^{\infty}\frac{1}{(n+r)^2......(1+r)^2}x^n]$$

$$\Rightarrow y_1 = y^*(x,r)\big|_{r=0} = a_0\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}$$

$$\Rightarrow y_2 = \frac{\partial y^*}{\partial r}\Big|_{r=0} = (\ln x)y_1 + a_0\sum_{n=1}^{\infty}\frac{-2}{(n+r)^2...(1+r)^2}[\frac{1}{n+r}+...+\frac{1}{1+r}]x^n\Big|_{r=0}$$

$$= (\ln x)y_1 + a_0\sum_{n=1}^{\infty}\frac{-2}{(n!)^2}(\frac{1}{n}+\frac{1}{n-1}+...+\frac{1}{1})x^n$$

Exponents differ by an integer

$Ly = 0$

$r_2 - r_1 = m$ ， $m \in Z$ （ $m \neq 0$ ）

$y_1 = y^*\big|_{r=r_1}$

y_2 ：分兩種情況討論

(1) $\lim_{r\to r_2} a_n$ 存在時， $y_2 = y^*\big|_{r=r_2}$

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③ $xy'' + (x-1)y' - y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \Rightarrow y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

紋的筆記-紋的筆記

$$\Rightarrow\ [r(r-1)-r]a_0x^{r-1}+\sum_{n=1}^{\infty}\{[(n+r)(n+r-1)-(n+r)]a_n+[(n+r-1)-1]a_{n-1}\}x^{n+r-1}=0$$

$$\Rightarrow\ r^2-2r=0\ \Rightarrow\ r=0,2$$

$$\Rightarrow\ r=0\ \Rightarrow\ a_n=-\frac{a_{n-1}}{n+r}=\frac{(-1)^n a_0}{(1+r)(2+r)......(n+r)}$$

$$\Rightarrow\ y^*=a_0x^r[1-\frac{x}{1+r}+\frac{x^2}{(1+r)(2+r)}-......+(\frac{(-1)^n x^n}{(1+r)(2+r)....(n+r)})+......]$$

$$\Rightarrow\ y_1=y^*|_{r=0}=a_0e^{-x}$$

$$y_2=y^*|_{r=2}=a_0x^2(1-\frac{x}{3}+\frac{x^2}{3\cdot 4}-......+\frac{(-1)^n x^n}{3\cdot 4\cdot ...\cdot(n+2)}+.....)$$

$$\Rightarrow\ y=c_1y_1+c_2y_2$$

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(2) $\lim\limits_{r\to r_2} a_n(r)$不存在，令$a_0=k(r-r_2)$得$a_n^*(r)$

$$y^*=\sum_{n=0}^{\infty}a_n^*x^{n+r}$$

$$y_2=\frac{\partial y^*}{\partial r}\Big|_{r=r_2}$$

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④ $x(1-x)y''+(2-5x)y'-4y=0$

Let $y=\sum\limits_{n=0}^{\infty}a_nx^{n+r}$

$$\Rightarrow\ [r(r+1)]a_0x^{r-1}+\sum_{n=1}^{\infty}(n+r-1)[(n+r)a_n-(n+r+1)a_{n-1}]x^{n+r-1}=0$$

$$\Rightarrow\ r(r+1)=0\ ，\ r=0,-1$$

$$\Rightarrow\ a_n=\frac{n+r+1}{n+r}a_{n-1}=\frac{n+r+1}{r+1}a_0$$

$$\Rightarrow\ y^*=a_0x^r(1+\frac{r+2}{r+1}x+\frac{r+3}{r+1}x^2+......+\frac{r+n+1}{r+1}x^n+...)$$

$$\Rightarrow\ y_1=y^*|_{r=0}=a_0(1+2x+3x^2+...+(n+1)x^n+...)$$

$\Rightarrow\ a_0=r+1$（因爲$\lim\limits_{r\to -1}a_n$不存在）

$$\Rightarrow\ y^*=x^r[(r+1)+(r+2)x+(r+3)x^2+......(r+n+1)x^n+...]$$

$$\Rightarrow\ y_2=y^*|_{r=-1}=x^{-1}(x+2x^2+3x^3+...)=1+2x+3x^2+...$$

（error，因爲$\lim\limits_{r\to -1}a_n$不存在）

$$\Rightarrow\ y_2=\frac{\partial y^*}{\partial r}\Big|_{r=-1}=\frac{\ln x}{x}(x+2x^2+...)+x^{-1}(1+x+x^2+...)$$

$$=\ln x(1+2x+3x^2+...)+\frac{1}{x}(1+x+x^2+...)$$

$$\Rightarrow\ y=c_1y_1+c_2y_2$$

The point at infinity

⑤ $2x^3\frac{d^2y}{dx^2}+3x^3\frac{dy}{dx}+y=0$

$\Rightarrow \frac{d^2y}{dx^2}+\frac{3}{2x}\frac{dy}{dx}+\frac{1}{2x^3}y=0$ ， $x=0$是 irregular point

$\Rightarrow x\equiv\frac{1}{t}$ ，x 在∞ is a regular singular point

$\Rightarrow y(t)=Y(t)$

$\Rightarrow \frac{dy}{dx}=\frac{dY}{dt}\frac{dt}{dx}=-\frac{1}{x^2}\frac{dY}{dt}=-t^2\frac{dY}{dt}$

$\Rightarrow \frac{d^2y}{dx^2}=t^4\frac{dY}{dt^2}+2t^3\frac{dY}{dt}$

$\Rightarrow 2t\frac{d^2Y}{dt^2}+\frac{dY}{dt}-Y=0$ ， $t=0$ is an regular singular point

$\Rightarrow r_1=\frac{1}{2}$ ， $r_2=0$

$\Rightarrow Y_1(t)=t^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!}t^n$

$Y_2(t)=\sum_{n=0}^{\infty}\frac{2^n}{(2n)!}t^n$

$\Rightarrow y_1=x^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!}x^{-n}$

$y_2=\sum_{n=0}^{\infty}\frac{2^n}{(2n)!}x^{-n}$

紋的筆記-★-紋的筆記

【綜合練習】

應用數學

① $3xy''+y'-y=0$ （85 交大光電）

Let $y=\sum_{n=0}^{\infty}a_nx^{n+r}$

$\Rightarrow 3\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}-\sum_{n=0}^{\infty}a_nx^{n+r}=0$

$$\Rightarrow [3r(r-1)+r]a_0x^{r-1}+\sum_{n=1}^{\infty}\{[3(n+r)(n+r-1)+(n+r)]a_n-a_{n-1}\}x^{n+r-1}=0$$

$$\Rightarrow 3r(r-1)+r=0 \Rightarrow r_1=0，r_2=\frac{2}{3}$$

$$\Rightarrow r_1=0 \Rightarrow a_n=\frac{1}{n(3n-2)}a_{n-1}$$

$$\Rightarrow y_1=1+x+\frac{1}{2\cdot 4}x^2+\frac{1}{2\cdot 3\cdot 4\cdot 7}x^3+\ldots\ldots$$

$$\Rightarrow r_2=\frac{2}{3} \Rightarrow a_n=\frac{1}{n(3n+2)}a_{n-1}$$

$$\Rightarrow y_2=x^{\frac{2}{3}}(1+\frac{1}{5}x+\frac{1}{80}x^2+\frac{1}{3\cdot 11\cdot 80}x^3+\ldots\ldots)$$

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② $xy''+y'+xy=0$

Let $y=\sum_{n=0}^{\infty}a_nx^{n+r}$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r+1}=0$$

$$\Rightarrow [r(r-1)+r]a_0x^{r-1}+(r+1)^2a_1x^r+\sum_{n=2}^{\infty}\{[(n+r)(n+r-1)+(n+r)]a_n+a_{n-1}\}x^{n+r-1}=0$$

$$\Rightarrow r=0,0$$

$$\Rightarrow a_n=-\frac{1}{(r+n)^2}a_{n-1}$$

$$\Rightarrow y_1=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\ldots$$

$$\Rightarrow y_2=\ln x(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\ldots)+(\frac{x^2}{2^2}-\frac{3x^4}{2\cdot 4^2}-\ldots)$$

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③ $2x^2y''+3xy'-(1+x)y=0$

Let $y=\sum_{n=0}^{\infty}a_nx^{n+r}$

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+3\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}=0$$

$$\Rightarrow [2r(r-1)+3r-1]a_0x^r+\sum_{n=1}^{\infty}\{[2(n+r)(n+r-1)+3(n+r)-1]a_n-a_{n-1}\}x^{n+r}=0$$

$$\Rightarrow (2r-1)(r+1)=0 \Rightarrow r=\frac{1}{2},-1$$

$$\Rightarrow r=\frac{1}{2} \Rightarrow a_n=\frac{1}{n(2n+3)}a_{n-1}=\frac{1}{n!5\cdot 7\cdot\ldots\cdot(2n+3)}a_0$$

$$\Rightarrow y_1=x^{\frac{1}{2}}[1+\sum_{n=1}^{\infty}\frac{x^n}{n!5\cdot 7\cdot\ldots\cdot(2n+3)}]$$

紋的筆記-☆-紋的筆記

$$\Rightarrow r=-1 \Rightarrow a_n=\frac{1}{n(2n-3)}a_{n-1}=\frac{1}{n!\cdot(-1)\cdot 1\cdot 3\cdot\ldots\cdot(2n-3)}a_0$$

$$\Rightarrow y_2=x^{-1}[1-x-\sum_{n=2}^{\infty}\frac{x^n}{n!\cdot 1\cdot 3\cdot\ldots\cdot(2n-3)}]$$

$$\Rightarrow y=c_1y_1+c_2y_2$$

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④ $(x^2-x)y''-xy'+y=0$ （84 成大電機）

Let $y=\sum_{n=0}^{\infty}a_nx^{n+r}$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}-\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\Rightarrow [-r(r-1)]a_0x^{r-1}+\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]a_nx^{n+r}-\sum_{n=1}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}=0$$

$$\Rightarrow [-r(r-1)]a_0x^{r-1}+\sum_{n=0}^{\infty}(n+r-1)^2a_nx^{n+r}-\sum_{n=0}^{\infty}(n+r)(n+r+1)a_{n+1}x^{n+r}=0$$

$$\Rightarrow [-r(r-1)]a_0x^{r-1}+\sum_{n=0}^{\infty}[(n+r-1)^2a_n-(n+r)(n+r+1)a_{n+1}]x^{n+r}=0$$

$$\Rightarrow r=0,1$$

$$a_{n+1}=\frac{(n+r-1)^2}{(n+r)(n+r+1)}a_n$$

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⑤ $x\frac{dy}{dx}-(x+1)y=x^2-x^3 \Rightarrow y'-\frac{x+1}{x}y=x-x^2$

$$\Rightarrow y_h=e^{\int(1+\frac{1}{x})dx}=e^{x+\ln x}=xe^x$$

$$\Rightarrow y=y_hV$$

$$\Rightarrow y_h'V+y_hV'-y_hV=x-x^2$$

$$\Rightarrow xe^xV'=x-x^2$$

$$\Rightarrow V'=e^{-x}-xe^{-x} \Rightarrow V=-e^{-x}+xe^{-x}+e^{-x}+c$$

$$\Rightarrow y=xe^x(xe^{-x}+c)=x^2+cxe^x$$

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⑥ $\frac{dy}{dx}+(\cot x)y=\sec x \Rightarrow y'+(\cot x)y=\sec x$

$$\Rightarrow y_h=e^{-\int\cot x dx}=e^{-\ln(\sin x)}=\frac{1}{\sin x}y$$

$$\Rightarrow y_h'V+y_hV'+y_hV=\sec x$$

$$\Rightarrow \frac{1}{\sin x}V'=\frac{1}{\cos x} \Rightarrow V'=\frac{\sin x}{\cos x} \Rightarrow V=-\ln|\cos x|+c$$

$$\Rightarrow y=\frac{c}{\sin x}-\frac{1}{\sin x}\ln|\cos x|$$

紋的筆記－紋的筆記

⑦ $(\tan^{-1}xy+\dfrac{y-2xy}{1+x^2y^2})dx+(\dfrac{x-2x^2}{1+x^2y^2})dy=0$

$\Rightarrow (\tan^{-1}xy)dx+\dfrac{1}{1+x^2y^2}[(ydx+xdy)-2x(ydx+xdy)]=0$

$\Rightarrow (\tan^{-1}xy)dx+\dfrac{1}{1+x^2y^2}(1-2x)d(xy)=0$

$\Rightarrow -\dfrac{1}{1-2x}dx=\dfrac{1}{\tan^{-1}xy}d(xy)\dfrac{1}{1+x^2y^2}$

$\Rightarrow \dfrac{1}{2}\ln(1-2x)=\ln(\tan^{-1}xy)+c$

$\Rightarrow -\ln(1-2x)+2\ln(\tan^{-1}xy)=d$　　（$d=-2c$）

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⑧ $x^2y''-3xy'+2y=6\ln x$

Let $\ln x=t \Rightarrow x=e^t$

$\Rightarrow \dfrac{dy}{dx}=e^{-t}\dfrac{dy}{dt}$，$\dfrac{d^2y}{dx^2}=e^{-2t}\dfrac{d^2y}{dt^2}$

$\Rightarrow y''-3y'+2y=6t$

$\Rightarrow \lambda=2,1$

$\Rightarrow y_h=c_1e^t+c_2e^{2t}$

$\Rightarrow y_p'=c_1e^t+2c_2e^{2t}+c_1'e^t+c_2'e^{2t}$　　$\Rightarrow c_1'e^t+c_2'e^{2t}=0$

$y_p''=c_1e^t+4c_2e^{2t}+c_1'e^t+2c_2'e^{2t}$　　$c_1'e^t+2c_2'e^{2t}=6t$

$\Rightarrow c_2'=6te^{-2t}$　　$\Rightarrow c_2=-3te^{-2t}-\dfrac{3}{2}e^{-2t}$

$c_1'=-6te^{-t}$　　$c_1=6te^{-t}+6e^{-t}$

$\Rightarrow y_p=6t+6-3t-\dfrac{3}{2}=3t+\dfrac{9}{2}$

$\Rightarrow y=c_1e^t+c_2e^{2t}+3t+\dfrac{9}{2}=c_1x+c_2x^2+3\ln x+\dfrac{9}{2}$

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⑨ $y''+2y'+y=x-6x^2e^{-x}+2e^{-x}$

$\Rightarrow \lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1,-1$

$\Rightarrow y_h=c_1e^{-x}+c_2xe^{-x}$

$\Rightarrow y_p=c_1(x)e^{-x}+c_2(x)xe^{-x}$

$\Rightarrow y_p'=-c_1e^{-x}+c_2e^{-x}-c_2xe^{-x}+c_1'e^{-x}+c_2'xe^{-x}$

$\Rightarrow y_p''=c_1e^{-x}-c_2e^{-x}-c_2e^{-x}+c_2xe^{-x}-c_1'e^{-x}+c_2'e^{-x}-c_2'xe^{-x}$

$$\Rightarrow \begin{cases} c_1'e^{-x}+c_2'xe^{-x}=0 \\ -c_1'e^{-x}+c_2'(e^{-x}-xe^{-x})=x-6x^2e^{-x}+2e^{-x} \end{cases}$$

$\Rightarrow c_2'e^{-x}=x-6x^2e^{-x}+2e^{-x}$

$\Rightarrow c_2'=xe^x-6x^2+2$　　$\Rightarrow c_2=xe^x-e^x-2x^3+2x$

紋的筆記

$c_1' = -x^2e^x - 6x^3 + 2x \qquad c_1 = -x^2e^x + 2xe^x - 2e^x + \frac{3}{2}x^4 - x^2$

$$\Rightarrow\ y = c_1e^{-x} + c_2xe^{-x} + (-x^2e^x + 2xe^x - 2e^x + \frac{3}{2}x^4 - x^2)e^{-x} + (xe^x - e^x - 2x^3 + 2x)xe^{-x}$$

$$= c_1e^{-x} + c_2xe^{-x} + (x-2) + (x^2 - \frac{1}{2}x^4)e^{-x}$$

---

⑩ $xy'' - 4xy' + 6y = x^4 \sin x \ \Rightarrow$ Let $y_h = x^\alpha$

$\Rightarrow\ \alpha(\alpha-1) - 4\alpha + 6 = 0 \ \Rightarrow\ \alpha = 2,3$

$\Rightarrow\ y_h = c_1x^2 + c_2x^3$

$\Rightarrow\ y_p = c_1(x)x^2 + c_2(x)x^3$

$\Rightarrow\ y_p' = 2c_1x + 3c_2x^2 + c_1'x^2 + c_2'x^3$

$\Rightarrow\ y_p'' = 2c_1 + 6c_2x + 2c_1'x + 3c_2'x^2$

$$\Rightarrow \begin{cases} c_1'x^2 + c_2'x^3 = 0 \\ x^2[2c_1'x + 3c_2'x^2] = x^4 \sin x \end{cases} \Rightarrow \begin{cases} c_1'x^2 + c_2'x^3 = 0 \\ 2c_1' + 3c_2'x = x\sin x \end{cases}$$

$\Rightarrow\ c_2' = \sin x \qquad \Rightarrow\ c_2 = -\cos x$

$c_1' = -x\sin x \qquad c_1 = x\cos x - \sin x$

$\Rightarrow\ y = c_1x^2 + c_2x^3 - x^2 \sin x$

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⑪ $xy'' + y' = \ln x \ \Rightarrow$ Let $u = y'$ ， $u' = y''$

$\Rightarrow\ xu' + u = \ln x$

$\Rightarrow\ u' + \frac{1}{x}u = \frac{1}{x}\ln x$

$\Rightarrow\ u_h = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

$\Rightarrow\ \frac{1}{x}V' = \frac{1}{x}\ln x$

$\Rightarrow\ V' = \ln x \ \Rightarrow\ V = x\ln x - x + c$

$\Rightarrow\ u = \ln x - 1 + \frac{c}{x} = y'$

$\Rightarrow\ y = x\ln x - 2x + c\ln x + c_1$

---

⑫ $(2x+1)^2 y'' - (12x+6)y' + 16y = 2$ （82 交大電子）

Let $2x+1 = t$

$\Rightarrow\ \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 2\frac{dy}{dt}$ ， $\frac{d^2y}{dx^2} = 4\frac{d^2y}{dt^2}$

$\Rightarrow\ 4t^2\frac{d^2y}{dt^2} - 12t\frac{dy}{dt} + 16y = 2$

$\Rightarrow\ t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 4y = \frac{1}{2}$

Let $y = t^\alpha$

紋的筆記

$$\Rightarrow\ \alpha(\alpha-1)-3\alpha+4=0\ \Rightarrow\ \alpha=2,2$$

$$\Rightarrow\ y_h=c_1t^2+c_2t^2\ln t$$

$$\Rightarrow\ y_p'=2c_1t+2c_2t\ln t+c_2t+c_1't^2+c_2't^2\ln t$$

$$\Rightarrow\ y_p''=2c_1+2c_2\ln t+2c_2+c_2+2c_1't+2c_2't\ln t+c_2't$$

$$\Rightarrow\ \begin{cases}c_1't^2+c_2't^2\ln t=0\\ 2c_1't^3+2c_2't^3\ln t+c_2't^3=\dfrac{1}{2}\end{cases}$$

$$\Rightarrow\ c_2'=\frac{1}{2}t^{-3}\qquad \Rightarrow\ c_2=-\frac{1}{4}t^{-2}\text{c}$$

$$c_1'=-\frac{1}{2}t^{-3}\ln t\qquad c_1=\frac{1}{4}t^{-2}\ln t+\frac{1}{8}t^{-2}$$

$$\Rightarrow\ y_p=\frac{1}{4}\ln t+\frac{1}{8}-\frac{1}{4}\ln t=\frac{1}{8}$$

$$\Rightarrow\ y_h=c_1t^2+c_2t^2\ln t+\frac{1}{8}$$

$$=c_1(2x+1)^2+c_2(2x+1)^2\ln|2x+1|+\frac{1}{8}$$

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⑬ $xy''=y'+(y')^3$　　$y(1)=2$ ， $y'(1)=1$

Let $y'=p$ ， $y''=p'$

$$\Rightarrow\ xp'=p+p^3$$

$$\Rightarrow\ x\frac{dp}{dx}=p+p^3$$

$$\Rightarrow\ \frac{dp}{p+p^3}=\frac{dx}{x}$$

$$\Rightarrow\ \ln\left|\frac{p}{\sqrt{1+p^2}}\right|=\ln|x|+c_1'$$

$$\Rightarrow\ \frac{p}{\sqrt{1+p^2}}=c_1x\ (\ c_1=e^{c_1'}\ )$$

Let $p=\tan\vartheta$ ， $1+p^2=\sec\vartheta$

$dp=\sec^2\vartheta\, d\vartheta$

$$\int\frac{1}{p(1+p^2)}dp=\int\frac{1}{\tan\vartheta}d\vartheta=\int\frac{\cos\vartheta}{\sin\vartheta}d\vartheta$$

$$=\ln|\sin\vartheta|=\ln\left|\frac{p}{\sqrt{1+p^2}}\right|$$

$$\Rightarrow\ p^2=c_1^2x^2(1+p^2)\qquad \Rightarrow\ (1-c_1^2x^2)p^2=c_1^2x^2$$

$$\Rightarrow\ p^2=\frac{c_1^2x^2}{1-c_1^2x^2}\qquad \Rightarrow\ p=\frac{c_1x}{\sqrt{1-c_1^2x^2}}=\frac{dy}{dx}$$

$$\Rightarrow\ p(1)=y'(1)=\frac{c_1}{\sqrt{1-c_1^2}}=1\qquad \Rightarrow\ c_1=\frac{1}{\sqrt{2}}$$

$$\Rightarrow\ y'=\frac{\frac{1}{\sqrt{2}}x}{\sqrt{1-\frac{1}{2}x^2}}=\frac{x}{\sqrt{2-x^2}}$$

$$\Rightarrow\ y=\int\frac{x}{\sqrt{2-x^2}}dx+c_2=-\sqrt{2-x^2}+c_2$$

$$\Rightarrow\ y(1)=2=-1+c_2\ \Rightarrow\ c_2=3$$

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$\Rightarrow \quad y = -\sqrt{2-x^2} + 3$

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⑭ $(D-2)^5 y = x^2 e^{2x}$

$\Rightarrow \quad y_h = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{2x}$

$\Rightarrow \quad y_p = \dfrac{1}{(D-2)^5} x^2 e^{2x} = e^{2x} \dfrac{1}{[(D+2)-2]^5} x^2 = e^{2x} \dfrac{1}{D^5} x^2$

$= e^{2x} \iiint\!\!\iint x^2 dx = \dfrac{1}{2520} x^7 e^{2x}$

$\Rightarrow \quad y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{2x} + \dfrac{1}{2520} x^7 e^{2x}$

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