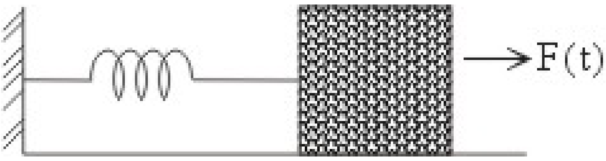
Simple Harmonic Motion：



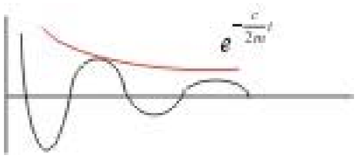
$m\ddot{x}+c\dot{x}+kx=F(x)$

Let $x_h=e^{\lambda t}$

1. $m\lambda^2+c\lambda+k=0$

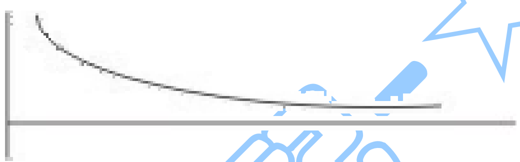
   $\Rightarrow\ \lambda=\dfrac{1}{2m}[-c\pm(c^2-2mk)^{\frac{1}{2}}]$

   ① $c^2<4mk$ ：兩共軛複數 underdamped



   ② $c^2>4mk$ ： $x_h(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$ ， $\lambda_1,\lambda_2\in R$

   overdamped， $\lambda_1,\lambda_2<0$



   ③ $c^2=4mk$ ： $x_h(t)=c_1e^{\lambda_1 t}+c_2te^{\lambda_2 t}$

---

2. $m\ddot{x}+kx=A\sin\omega t$

   $x(t)=c_1\cos\omega t+c_2\sin\omega t+\dfrac{A}{m(\omega^2-\omega_1^2)}\sin\omega t$

   若 $\omega\approx\omega_1$ （ $\omega^2-\omega_1^2=(\omega+\omega_1)(\omega-\omega_1)$ ）

   $x(t)=c_1\cos\omega t+c_2\sin\omega t-\dfrac{A}{2m\omega}\cos\omega t$ ………resonance

Electric circule：

$$V(t)-RI-\frac{Q}{C}-L\frac{dI}{dt}=0$$

$$\Rightarrow\ L\frac{d^2I}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=V(t)$$

Gamma function：

$\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt$ ，$\nu>0$

$\Gamma(1)=\int_0^\infty e^{-t}dt=-e^{-t}\Big|_0^\infty=1$

$\Gamma(\nu+1)=\nu\Gamma(\nu)$

pf： $\Gamma(\nu+1)=\int_0^\infty e^{-t}t^{\nu}dt$

$$=t^{\nu}(-e^{-t})\Big|_0^\infty-\int_0^\infty \nu\cdot t^{\nu-1}(-e^{-t})dt$$

$$=\nu\Gamma(\nu)$$

ν 是自然數，$\nu=n$

$\Gamma(n+1)=n\Gamma(n)$

$$=n(n-1)\Gamma(n-1)$$

$$=n!\Gamma(1)$$

$$=n!$$

$0!=1$ ，$0!=\Gamma(1)$ ，$\Gamma(\tfrac{1}{2})=\sqrt{\pi}$

pf： $\Gamma(\tfrac{1}{2})=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$

$$\overset{t=u^2}{=}\int_0^\infty e^{-u^2}u^{-1}2udu$$

$$=2\int_0^\infty e^{-u^2}du$$

（ $\int_{-\infty}^0 e^{-u^2}du\overset{u=-u}{=}\int_\infty^0 e^{-u^2}(-du)=\int_0^\infty e^{-u^2}du$ ）

$[\Gamma(\tfrac{1}{2})]^2=(\int_{-\infty}^\infty e^{-u^2}du)(\int_{-\infty}^\infty e^{-v^2}dv)$

$$=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(u^2+v^2)}dudv \qquad u=r\cos\vartheta\text{，}v=r\sin\vartheta\text{，}u^2+v^2=r^2$$

$$=\int_0^{2\pi}\int_0^\infty e^{-r^2}rdrd\vartheta$$

$$=2\pi\cdot\frac{1}{2}=\pi$$

$\therefore\ \Gamma(\tfrac{1}{2})=\sqrt{\pi}$

$\Gamma(\tfrac{3}{2})=\dfrac{1}{2}\Gamma(\tfrac{1}{2})=\dfrac{\sqrt{\pi}}{2}$

$\Gamma(\tfrac{5}{2})=\dfrac{3}{2}\cdot\dfrac{1}{2}\Gamma(\tfrac{1}{2})=\dfrac{3}{4}\sqrt{\pi}$

$$\int_0^\infty e^{-s^3}ds = \int_0^\infty e^{-t}(\frac{1}{3}t^{-\frac{2}{3}}dt)$$
$$= \frac{1}{3}\int_0^\infty e^{-t}t^{\frac{1}{3}-1}dt = \frac{1}{3}\Gamma(\tfrac{1}{3})$$

Bessel's Equation：

$x^2y'' + xy' + (x^2 - v^2)y = 0$（$v$ 爲常數）

$x = 0$ 是 regular singular point

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - v^2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow [r^2 - v^2]a_0x^r + [(r+1)r + r + 1 - v^2]a_1x^{r+1} + \sum_{n=2}^{\infty}\{[(n+r)(n+r-1) + (n+r) - v^2]a_n + a_{n+2}\}x^{n+r} = 0$$

$\Rightarrow r^2 - v^2 = 0 \Rightarrow r = \pm v \Rightarrow r = r_1 = v$ 代入 x^{r+1} 的係數

$\Rightarrow (r+1)^2 - v^2 = 0$ 不合（$\because (r+1)^2 = v^2$）

$\Rightarrow a_1 = 0 \Rightarrow$ 只剩下 a_n，n 是偶數

$\Rightarrow [(n+v)(n+v-1) + (n+v) - v^2]a_n + a_{n+2} = 0$

$$\Rightarrow (n^2 + 2nv)a_n + a_{n-2} = 0 \Rightarrow a_n = -\frac{1}{n(n+2v)}a_{n-2}$$

$$\Rightarrow a_n = [-\frac{1}{n(n+2v)}][-\frac{1}{(n-2)(n-2+2v)}]a_{n-4}$$
$$= \frac{(-1)^m}{2^{2m}m!(1+v)\dots(m+v)}a_0 \text{……} n \text{ 是偶數}$$

$n!! = n(n-2)(n-4)\dots2$

$\overset{n=2m}{=} 2m(2m-2)(2m-4)\dots2$

$= 2^m \cdot m!$

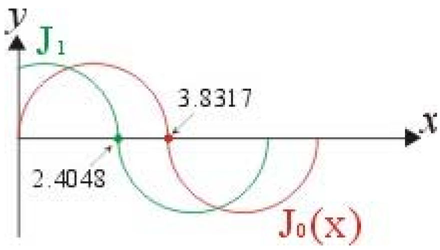
$[(n+2v)(n+2v-2)\dots] \overset{n=2m}{=} (2m+2v)(2m+2v-2)\dots.(2v+2)$

$= 2^m(m+v)(m+v-1)\dots(v+1)$

$a_n = \dfrac{(-1)^m \cdot v!}{2^{2m}(m+v)!}a_0$（$m = 1,2,3,\dots$）

選擇：$a_0 = \dfrac{1}{2^v \cdot v!} = \dfrac{1}{2^v\Gamma(1+v)}$

$$y_1 = J_v(x) = \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{x}{2})^{2m+v}}{m!\Gamma(m+v+1)}$$ ……（Bessel function of the first kind of order v）



① $\Gamma(x)=\int_0^{\infty} e^{-t}t^{x-1}dt$

② $\Gamma(x+1)=x\Gamma(x)$

③ $\Gamma(1)=1$

④ $\Gamma(\frac{1}{2})=\sqrt{\pi}$

$\Rightarrow\ \Gamma(m+v+1)=(m+v)!$

$\Rightarrow\ \Gamma(n+1)=n!$

$r=-v\ \Rightarrow\ y_2=J_{-v}(x)$ ，$v\neq$ 整數

$\Rightarrow\ y=c_1J_v(x)+c_2J_{-v}(x)$

若 v 是整數

$$J_{-v}(x)=(-1)^vJ_v(x)$$

$$J_{-n}=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{x}{2})^{2m-n}}{m!\Gamma(m-n+1)}$$

$$\frac{1}{\Gamma(m-n+1)}=0\quad \text{當 } m=0,1,2,\ldots,(n-1)$$

$$\Rightarrow\ m-n+1=0$$

$$\Rightarrow\ m=n-1$$

$$\Rightarrow\ \Gamma(0)\to\infty$$

$$\Rightarrow\ \frac{1}{\Gamma(0)}\to 0$$

$$J_{-v}(x)=\sum_{m=n}^{\infty}\frac{(-1)^m(\frac{x}{2})^{2m-n}}{m!\Gamma(m-n+1)}$$

$$\overset{m\to s+n}{=}\sum_{s=0}^{\infty}\frac{(-1)^{s+n}(\frac{x}{2})^{2s+n}}{(s+n)!\Gamma(s+n-n+1)}$$

$$=(-1)^n\sum_{s=0}^{\infty}\frac{(-1)^s(\frac{x}{2})^{2s+n}}{\Gamma(s+n+1)s!}$$

$$=(-1)^nJ_n(x)$$

$v=n$ ，$J_{-n}(x)$ 和 $J_n(x)$ 線性相關

$\Rightarrow$ 要求 second solution

Neumann function N_v ：

$$Yv(x)=\frac{J_v\cos v\pi-J_{-v}(x)}{\sin v\pi}\underset{r\to n}{\to}\frac{0}{0}$$

v 是整數， $\sin n\pi=0$ ， $\cos n\pi=(-1)^n$

$J_{-n}(x)=(-1)^nJ_n(x)$

$Y_v(x)\underset{x\to 0}{\to}-\infty$

ex： $x^2y''+xy'+(x^2-v^2)y=0 \Rightarrow y=c_1J_v(x)+c_2N_v(x)$

比較 $\ddot{y}+\omega^2y=0$

$\Rightarrow y=c_1\cos\omega t+c_2\sin\omega t$

Hankel functions（Bessel function of the third kind）：

$$H_v^{(1)}(x)=J_v(x)+iY_v(x)$$
$$H_v^{(2)}(x)=J_v(x)-iY_v(x)$$
$$J_v(x)\underset{x\to\infty}{\to}\frac{1}{\sqrt{x}}\cos(...)$$
$$N_v(x)\underset{x\to\infty}{\to}\frac{1}{\sqrt{x}}\sin(...)$$
$$H_v^{(1)}(x)\underset{x\to\infty}{\to}\frac{1}{\sqrt{x}}e^{i(...)}$$

1. $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$
2. $J_n'(x)=\frac{1}{2}[J_{n-1}(x)-J_{n+1}(x)]$
3. $xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$
4. $xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)$
5. $\frac{d}{dx}[x^nJ_n(x)]=x^nJ_{n-1}(x)$
6. $\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x)$

Bessel 的性質：

Generating function（生成函數）

$$F(x,t)\equiv e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n$$

$$F(x,t)\equiv e^{\frac{x}{2}t}\cdot e^{-\frac{x}{2t}}=[\sum_{r=0}^{\infty}(\frac{x}{2})^r\frac{t^r}{r!}][\sum_{s=0}^{\infty}(-1)^s(\frac{x}{2})^s\frac{t^{-s}}{s!}]$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(\frac{x}{2})^{r+s}}{r!s!}t^{r-s}$$

$$=\sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(\frac{x}{2})^{2s+n}}{(s+n)!s!}t^n \qquad \left(\begin{matrix}r-s=n，r\ge 0，s\ge 0\\ r=s+n，n=r-s，-\infty<n<\infty\end{matrix}\right)$$

$$=\sum_{n=-\infty}^{\infty}J_n(x)t^n$$

$$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s+n)!}(\frac{x}{2})^{2s+n}$$

Recurrence relation：

① $F(x,t)$對 t 微分 $\Rightarrow\ J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$

$$F(x,t)\equiv e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=0}^{\infty}J_n(x)t^n \quad = e^{\frac{x}{2}(t-\frac{1}{t})}\cdot(\frac{x}{2})\cdot(1+\frac{1}{t^2})=\sum_{n=-\infty}^{\infty}J_n(x)nt^{n-1}$$

$$\Rightarrow\ \sum_{n=-\infty}^{\infty}J_n(x)t^n+\sum_{n=-\infty}^{\infty}J_n(x)t^{n-2}=\frac{x}{2}\sum_{n=-\infty}^{\infty}J_n(x)nt^{n-1}$$

$\Rightarrow\ t^{n-1}$係數

$$\Rightarrow\ J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$$

---

② $F(x,t)$對 x 微分 $\Rightarrow\ J_{n-1}(x)-J_{n+1}(x)=2J_n'(x)$

$$\frac{d}{dx}[x^vJ_v(x)]=x^vJ_{v-1}(x)$$

$$pf：\frac{d}{dx}[x^vJ_v(x)]=\frac{d}{dx}\sum_{m=0}^{\infty}\frac{(-1)^m(x)^{2m+2v}}{2^{2m+v}\cdot m!\Gamma(m+v+1)}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+2v-1}}{2^{2m+v-1}\cdot m!\Gamma(m+v)}$$

$$=x^vJ_{v-1}(x)$$

$$\frac{d}{dx}[x^{-v}J_v(x)]=-x^{-v}J_{v+1}(x)$$

---

ex：Fraunhofer Diffraction

$$\Phi=\int_0^a\int_0^{2\pi}e^{ibr\cos\vartheta}rd\vartheta dr=\int_0^a 2\pi J_0(br)rdr$$

$$=\frac{2\pi}{b^2}\int_0^a J_0(x)xdx$$

$$=\frac{2\pi}{b^2}\int_0^a\frac{d}{dx}[xJ_1(x)]dx$$

$$=\frac{2\pi}{b^2}xJ_1(x)\Big|_0^a$$

$$=\frac{2\pi}{b^2}aJ_1(ab)\Big|_0^a$$

Legendre's equation：

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

$\Rightarrow\ x=\pm1$有 single point

Let $y=\sum_{m=0}^{\infty}a_mx^m$

$$\Rightarrow\ (1-x^2)\sum_{m=2}^{\infty}m(m-1)a_mx^{m-2}-2x\sum_{m=1}^{\infty}ma_mx^{m-1}+n(n+1)\sum_{m=0}^{\infty}a_mx^m=0$$

x_0 係數 $\Rightarrow\ 2a_2+n(n+1)a_0=0$

x_1 係數 $\Rightarrow\ 6a_3+[-2+n(n+1)]a_1=0$

$$\Rightarrow\ a_{m+2}=-\frac{(n-m)(n+m+1)}{(m+2)(m+1)}a_m \quad (m=0,1,2,.....)$$

$$y_1(x)=1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4+......$$

$$y_2(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+3)(n+4)}{5!}x^5+......$$

$$n=0\ \Rightarrow\ y_2=x+\frac{x^3}{3!}+\frac{x^5}{5!}+......=\frac{1}{2}\ln\frac{1+x}{1-x}$$

$$\begin{cases} x=1 & y_2\to\infty \\ x=-1 & y_2\to-\infty \end{cases}$$

解決方法：n 取整數

$n=m$ 時，級數會有截斷（cut off）

級數會變成多項式

Legendre's polynomials (functions)

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}(3x^2-1)$$

$$P_3(x)=\frac{1}{2}(5x^3-3x)$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

$$P_4(x)=\frac{1}{8}(63x^5-70x^3+15x)$$

性質：(1) $P_{2m}(-x)=P_{2m}(x)$ ………………even function

$P_{2m+1}(-x)=-P_{2m+1}(x)$ …………..odd function

(2) $P_n(x)=\dfrac{1}{2^n\cdot n!}\dfrac{d^n}{dx^n}[x^2-1]^n$ ……..Rodrigue's formula

(3) orthogonal（正交）

$$\int_{-1}^{1}P_n(x)P_{n'}(x)dx=\frac{2}{2n+1}S_{nn'}$$

(4) generating function

$$F(x,n)=(1-2xu+u^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)u^n$$

Let $t=2xu-u^2$

$$(1-t)^{-\frac{1}{2}}=1-\frac{1}{2}(-t)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-t)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-t)^3+\ldots$$

$$=1+\frac{1}{2}t+\frac{3}{8}t^2+\ldots\ldots$$

$$=1+\frac{1}{2}(2xu-u^2)+\frac{3}{8}(2xu-u^2)^2+\ldots\ldots$$

$$F(x,n)=1+\sum_{m=1}^{\infty}\frac{(-\frac{1}{2})(-\frac{1}{2}-1)\ldots\ldots(-\frac{1}{2}-m+1)}{m!}(-t)^m$$

$$=1+\sum_{m=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2m-1)}{2^m\cdot m!}(t)^m$$

$$=1+\sum_{m=1}^{\infty}\frac{(2m)!}{2^m\cdot(m!)^2}(t)^m$$

$$t^m=(2xu-u^2)^m \qquad (x+y)^m=\sum_{n=0}^{m}\binom{m}{n}x^{m-n}y^n$$

$$=\sum_{j=0}^{m}\binom{m}{j}(2xu)^{m-j}(-u^2)^j \qquad \binom{m}{n}=\frac{m!}{(m-n)!n!}$$

$$=\sum_{j=0}^{m}\frac{(-1)^j\cdot m!\cdot(2x)^{m-j}}{j!\cdot(m-j)!}u^{m+j}$$

$$F(x,u)=\sum_{m=0}^{\infty}\left[\sum_{j=0}^{m}\frac{(-1)^j\cdot(2m)!\cdot(2x)^{m-j}}{2^{2m}\cdot m!\cdot j!\cdot(m-j)!}u^{m+j}\right] \qquad (\ m=n-k\ ，\ j=k\ )$$

$$=\sum_{n=0}^{\infty}P_n(x)u^n$$

Let $n=m+j$ ， $j=k\geq 0$ ， $m=n-k\geq 0$

$$F(x,u)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k\cdot(2n-2k)!\cdot x^{n-2k}}{2^n\cdot(n-k)!\cdot k!\cdot(n-2k)!}u^n\right]$$

$0\leq k\leq n-k$ （$\because m-j\geq 0$）， $n-k\geq 0$ ， $0\leq 2k\leq n$ ， $k\leq[\frac{n}{2}]$

---

ex： $V(\vec{r})=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int\frac{e(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'$

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\frac{1}{[(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')]^{\frac{1}{2}}}=(r^2+r'^2-2\vec{r}\cdot\vec{r}\,')^{-\frac{1}{2}}$$

$$=\frac{1}{r[1+\frac{r'^2}{r^2}-\frac{2rr'\cos\vartheta}{r^2}]^{\frac{1}{2}}} \quad ，\ u\equiv\frac{r'}{r} \ ，\ x=\cos\vartheta$$

$$= \frac{1}{r}\frac{1}{(1-2xu+u^2)^{\frac{1}{2}}} \qquad \left(F(x,u) = \frac{1}{(1-2xu+u^2)^{\frac{1}{2}}} \right)$$

$$= \frac{1}{r}\sum_{n=0}^{\infty} P_n(x)u^n$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\vec{r}\,')\sum_{n=0}^{\infty} P_n(x)u^n d^3r'$$