Laplace Transform

$F(s)=\int_0^\infty e^{-st} f(t)dt = L[f](s)$

函數 $f(t)$ 的Laplace變換爲 $F(s)$

取適當s，使 $F(s)$ 收斂

$L^{-1}[F](t) = f(t)$ （L^{-1} ：inverse Laplace Transform）

---

ex： $f(t)=e^{t^2} \Rightarrow F(s)=\int_0^\infty e^{t^2}\cdot e^{-st}dt$ 不收斂

---

ex： $f(t)=1 \Rightarrow F(s)=\int_0^\infty e^{-st}\cdot 1\cdot dt$

$= (-\frac{1}{s})e^{-st}\Big|_0^\infty = \frac{1}{s}$ （ $s>0$ ）

---

ex： $f(t)=e^{at} \Rightarrow F(s)=\int_0^\infty e^{-st}\cdot e^{at}dt$

$=\int_0^\infty e^{-(s-a)t}dt$

$=\frac{1}{s-a}$ （ $s>a$ ）

---

ex： $f(t)=\cos at \Rightarrow F(s)=\int_0^\infty e^{-st}\cdot \cos at\cdot dt$

$=\frac{1}{2}\int_0^\infty e^{-st}[e^{iat}+e^{-iat}]dt$

$=\frac{1}{2}[\int_0^\infty e^{-(s-ia)t}dt+\int_0^\infty e^{-(s+ia)t}dt]$

$=[\frac{1}{s-ia}+\frac{1}{s+ia}]$

$=\frac{1}{2}\frac{2s}{s^2+a^2}=\frac{s}{s^2+a^2}$

---

ex： $f(t)=\sin at \rightarrow \frac{e^{iat}-e^{-iat}}{2i} \Rightarrow F(s)=\frac{a}{s^2+a^2}$

$\cosh at = \frac{e^{at}+e^{-at}}{2}$

$\sinh at = \frac{e^{at}-e^{-at}}{2}$

$\cosh^2 at = 1+\sinh^2 at$

$\tanh at = \frac{\sinh at}{\cosh at}$

ex： $f(t)=\cosh at \Rightarrow F(s)=\dfrac{s}{s^2-a^2}$

---

ex： $f(t)=\sinh at \Rightarrow F(s)=\int_0^\infty e^{-st}\cdot\sinh at\cdot dt$

$$=\int_0^\infty e^{-st}\cdot\frac{e^{at}-e^{-at}}{2}\cdot dt$$

$$=\frac{1}{2}\int_0^\infty [e^{-(s-a)t}-e^{-(s+a)t}]dt$$

$$=\frac{1}{2}(\frac{1}{s-a}-\frac{1}{s+a})=\frac{a}{s^2-a^2}$$

---

ex： $f(t)=t^n \Rightarrow F(s)=\int_0^\infty e^{-st}\cdot t^n\cdot dt$

$$=\frac{1}{s^{n+1}}\int_0^\infty e^{-y}y^n dy$$

$$=\frac{1}{s^{n+1}}\Gamma(n+1)$$

$$=\frac{1}{s^{n+1}}\cdot n!$$

---

ex： $f(t)=t^{-\frac{1}{2}} \Rightarrow F(s)=\int_0^\infty e^{-st}\cdot t^{-\frac{1}{2}}\cdot dt$

$$=\int_0^\infty e^{-y}\cdot s^{\frac{1}{2}}\cdot y^{-\frac{1}{2}}\cdot\frac{1}{s}dy$$

$$=s^{-\frac{1}{2}}\Gamma(\frac{1}{2})=\sqrt{\frac{\pi}{s}}$$

應用數學

Transforms of Derivatives

設 $f(t)$ 、 $f'(t)$ 在 $s>s_0$ 時，Laplace transform存在

$$\int_0^\infty e^{-st}f'(t)dt=F(s)$$

$$=e^{-st}f(t)\Big|_0^\infty-\int_0^\infty(-se^{-st})f(t)dt$$

$$=-f(0)+sF(s)$$

---

ex： $x'(t)+2x(t)=e^{-t}$ ， $x(0)=2$

$$\Rightarrow sX(s)-x(0)+2X(s)=\frac{1}{s+1}$$

$$\Rightarrow (s+2)X(s)=2+\frac{1}{s+1}$$

$$\Rightarrow X(s)=\frac{2s+3}{(s+1)(s+2)}=\frac{1}{s+1}+\frac{1}{s+2}$$

$\Rightarrow\ s(t) = L^{-1}[X(s)]$

$\Rightarrow\ x(t) = e^{-t} + e^{-2t}$

設 $f(t)$、$f'(t)$、$f''(t)$、……、$f^{(n-1)}(t)$ 是連續函數

$$L[f^{(n)}](s) = s^n F(s) - [s^{n-1} f(0) + s^{n-2} f'(0) + \dots\dots + f^{(n-1)}(0)]$$

pf： $L[f'] = sF(s) - f(0)$

若 $m = n$ 成立

$$L[f^{(m+1)}](s) = \int_0^\infty e^{-st} f^{(m+1)}(t)dt$$

$$= e^{-st} f^{(m)}(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f^{(m)}(t)dt$$

$$= -f^{(m)}(0) + sL[f^{(m)}]$$

……..

$$L[f'] = sF(s) - f(0)$$

$$L[f''] = s^2 F(s) - [sf(0) + f'(0)]$$

---

ex： $f(t) = t^2$，$f(0) = 0$

$\Rightarrow\ f'(t) = 2t$，$f'(0) = 0$

$\Rightarrow\ f''(t) = 2$

$\Rightarrow\ L[f''(t)] = L[2] = \dfrac{2}{s}$

$$= s^2 F(s) - [sf(0) + f'(0)] = s^2 \cdot \frac{2}{s^3} = \frac{2}{s}$$

（ $F(s) = L[t^2] = \dfrac{2}{s^3}$ ）

Integral of a function

$$L[\int_0^t f(\tau)d\tau] = \frac{1}{s}F(s) \qquad g(t) = \int_0^t f(\tau)d\tau$$

pf： $L[g(t)] = \int_0^\infty e^{-st} g(t)dt = -\dfrac{1}{s}e^{-st} g(t)\Big|_0^\infty - \int(-\dfrac{1}{s}e^{-st})f(t)dt = \dfrac{1}{s}F(s)$

---

ex： $f(t) = \int_0^t \sin 2u du \ \Rightarrow\ L[f] = \dfrac{1}{s}(\dfrac{2}{s^2+4})$

---

ex： $L[f] = \dfrac{1}{s(s^2+\omega^2)}$，find $f(t)$

$\Rightarrow\ L[f] = \dfrac{1}{s(s^2+\omega^2)} = (\dfrac{1}{s}\cdot\dfrac{\omega}{s^2+\omega^2})\dfrac{1}{\omega}$

$\Rightarrow\ f(t) = \dfrac{1}{\omega}\int_0^t \sin\omega\tau d\tau$

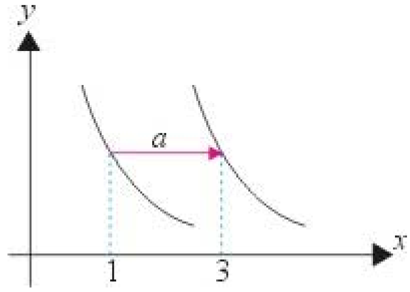
$$= \frac{1}{\omega}(-\frac{1}{\omega}\cos\omega\tau)\Big|_0^t \qquad (\ L[af] = \int_0^\infty e^{-st} af(t)dt = aF(s)\ )$$

$$= \frac{1}{\omega^2}(1-\cos\omega t)$$

---

s-shifting……s空間，圖形 $F(s)$ 有位移

$$L[e^{at} f(t)] = F(s-a)$$



圖往右走

$F(S) \to f(S-A)$

1　　3　2

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a)$$

---

ex： $f(t)$ ，find $L[f]$

$$\Rightarrow\ e^{at}t^n \quad \Rightarrow \quad \frac{n!}{(s-a)^{n+1}}$$

$$e^{at}\cos\omega t \quad \Rightarrow \quad \frac{s-a}{(s-a)^2+\omega^2}$$

$$e^{at}\sin\omega t \quad \Rightarrow \quad \frac{\omega}{(s-a)^2+\omega^2}$$

---

ex： $y''+2y'+5y=0$ ， $y(0)=2$ ， $y'(0)=-4$

$$\Rightarrow\ s^2Y-(2s-4)+2(sY-2)+5Y=0$$

$$(\ Y(s) = \int_0^\infty e^{-st} y(t)dt\ )$$

$$\Rightarrow\ (s^2+2s+5)Y = 2s$$

$$\Rightarrow\ Y = \frac{2s}{s^2+2s+5} = \frac{2(s+1)}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$\Rightarrow\ y(t) = L^{-1}[Y] = 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

---

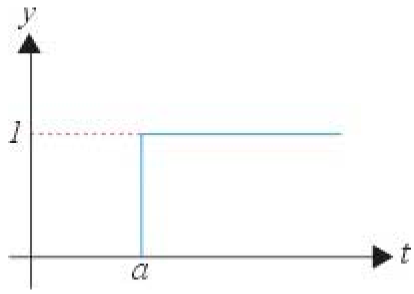
t-shifting

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

$$\Leftrightarrow\ L^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$$

step function

$$u(t-a)=\begin{cases}0 & t<a\\ 1 & t\ge a\end{cases}$$



$$pf：F(s)=\int_0^\infty e^{-st}f(t)dt \;\Rightarrow\; e^{-as}F(s)=e^{-as}\int_0^\infty e^{-st}f(t)dt$$

$$=\int_0^\infty e^{-(t+a)s}f(t)dt$$

$$\overset{t=\tau-a}{=}\int_0^\infty e^{-\tau s}f(\tau-a)d\tau$$

$$=\int_0^\infty e^{-\tau s}f(\tau-a)u(\tau-a)d\tau$$

$$=L[f(t-a)u(t-a)]$$

$$\text{ex：}L[u(t-a)]=\frac{1}{s}e^{-as} \;\Rightarrow\; \int_0^\infty e^{-st}u(t-a)dt=\int_a^\infty e^{-st}dt$$

$$=-\frac{1}{s}e^{-st}\Big|_a^\infty=\frac{1}{s}e^{-sa}$$

$$\text{ex：}L^{-1}[\frac{e^{-3s}}{s^3}] \;\Rightarrow\; L^{-1}[\frac{1}{s^3}]=\frac{t^2}{2}$$

$$L^{-1}[\frac{e^{-3s}}{s^3}]=\frac{1}{2}(t-3)^2\cdot u(t-3)=\begin{cases}0 & t<3\\ \frac{1}{2}(t-3)^2 & t>3\end{cases}$$

$$(\,L[t^n]=\frac{n!}{s^{n+1}}\,)$$

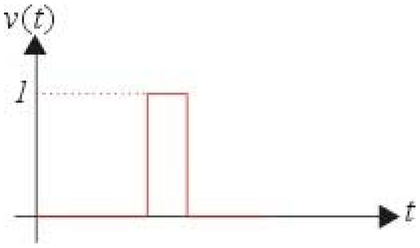
$$\text{ex：}f(t)=\begin{cases}2 & 0<t<\pi\\ 0 & \pi<t<2\pi\\ \sin t & t>2\pi\end{cases}$$

$$\Rightarrow L[f]=F(s)$$

$$\Rightarrow f(t)=2u(t)-2u(t-\pi)+u(t-2\pi)\sin t$$

$$\Rightarrow L[f]=\frac{2}{s}-\frac{2e^{-\pi s}}{s}+\frac{e^{-2\pi s}}{s^2+1}$$

$$(\,\int_0^\infty e^{-st}u(t)dt=\int_0^\infty e^{-st}dt=\frac{1}{s}\,)$$



pluse脈衝

$$v(t)=u(t-t_1)-u(t-t_2)$$

---

ex： $F(s)=\dfrac{1}{s^2}-\dfrac{2e^{-2s}}{s^2}-\dfrac{4e^{-2s}}{s}+\dfrac{se^{-\pi s}}{s^2+1}$ ， $f(t)=?$

$$\Rightarrow\ f(t)=L^{-1}[F]$$
$$=t-2(t-2)u(t-2)-4u(t-2)+\cos(t-\pi)u(t-\pi)$$

應用數學

Dirac's Delta Function

$$\delta(t-a)=\begin{cases}\infty & t=a\\ 0 & t\neq a\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(t-a)dt=1\ \Rightarrow\ \int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$$

ex：fore $F(t)=\delta(t)$

$$F=\frac{dP}{dt}\ ,\ P=\int Fdt=\int_{-\infty}^{\infty}\delta(t)dt=1$$

---

ex： $\int_0^2\delta(t-3)dt=0$

---

ex： $y''+3y'+2y=\delta(t-a)$ ， $y(0)=0$ ， $y'(0)=0$

$$\overset{Y(s)=L[y]}{\Rightarrow}\ s^2Y+3sY+2Y=e^{-sa}\qquad \left(\int_0^{\infty}e^{-st}\delta(t-a)dt=e^{-sa}\right)$$

$$\Rightarrow\ Y(s)=\frac{e^{-sa}}{(s+1)(s+2)}=e^{-sa}\cdot F(s)\qquad \left(F(s)=\frac{1}{(s+1)(s+2)}\right)$$

$$\Rightarrow\ f(t)=e^{-t}-e^{-2t}$$

$$\Rightarrow\ y(t)=L^{-1}\{e^{-sa}F(s)\}$$
$$=f(t-a)u(t-a)$$
$$=[e^{-(t-a)}-e^{-2(t-a)}]u(t-a)$$
$$=\begin{cases}0 & 0\le t<a\\ e^{-(t-a)}-e^{-2(t-a)} & t>a\end{cases}$$

應用數學

Differentiation of transform

$$F(s)=\int_0^{\infty}e^{-st}f(t)dt$$

$$F'(s) = -\int_0^\infty e^{-st} t \cdot f(t)dt = -L[tf(t)]$$

$$\Rightarrow \boxed{L[tf(t)] = -F'(s)} \quad \text{or} \quad \boxed{L^{-1}[F(s)] = -tf(t)}$$

ex： $L[t\sin\beta t] = -(\frac{\beta}{s^2+\beta^2})' = \frac{2\beta s}{(s^2+\beta^2)^2}$

---

ex： $L[t\cos\beta t] = -(\frac{s}{s^2+\beta^2})' = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$

$$\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)}$$

應用數學

Integration of transforms

$$L[\frac{f(t)}{t}] = \int_s^\infty F(\tilde{s})d\tilde{s}$$

pf： $\int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty [\int_0^\infty e^{-\tilde{s}t} f(t)dt]d\tilde{s}$

$$= \int_0^\infty (\int_s^\infty e^{-\tilde{s}t} d\tilde{s}) f(t)dt$$

$$= \int_0^\infty (-\frac{1}{t} e^{-\tilde{s}t}\Big|_0^\infty) f(t)dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)dt = L[\frac{f(t)}{t}]$$

---

ex： $L^{-1}[\ln(1+\frac{\omega^2}{s^2})] = -\frac{d}{ds}\ln(1+\frac{\omega^2}{s^2}) = \frac{\frac{2\omega^2}{s^2}}{1+\frac{\omega^2}{s^2}} = \frac{2\omega^2}{s(s^2+\omega^2)} \; F(s)$

$$F(s) = \frac{a}{s} + \frac{b+c}{s^2+\omega^2} = \frac{2}{s} - \frac{2s}{s^2+\omega^2}$$

$\therefore \quad f(t) = 2 - 2\cos\omega t$

$\therefore L^{-1}[\ln(1+\frac{\omega^2}{s^2})] = \frac{2-2\cos\omega t}{t}$

---

ex：Legendre's differential equation $ty'' + (1-t)y' + ny = 0$ ， $y = ?$

$$\Rightarrow -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] + sY - y(0) - (-\frac{d}{ds})[sY - y(0)] + nY = 0$$

$$\Rightarrow -2sY - s^2Y' + y(0) + sY - y(0) + [Y + sY'] + nY = 0$$

$$\Rightarrow (s-s^2)Y' + (n+1-s)Y = 0$$

$$\Rightarrow \frac{dY}{Y} = \frac{n+1-s}{s-s^2} ds = (\frac{n}{s-1} - \frac{n+1}{s})ds$$

$$\Rightarrow \ln Y = n\ln(s-1)-(n+1)\ln s+\alpha$$

$$\Rightarrow Y = A\cdot\frac{(s-1)^n}{s^{n+1}} \qquad (A=1)$$

$$\Rightarrow y = L^{-1}[Y] = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n\cdot e^{-t})$$

$$L[t^n e^{-t}] = (-1)^n\frac{d^n}{ds^n}(\frac{1}{s+1}) = \frac{n!}{(s+1)^{n+1}}$$

$$L[(t^n e^{-t})^{(n)}] = s^n\frac{n!}{(s+1)^{n+1}}$$

$$L[e^t(t^n e^{-t})^n] = \frac{(s-1)^n n!}{s^{n+1}}$$

應用數學

Convolution（卷積）

$$f*g = \int_0^t f(t-\tau)g(\tau)d\tau$$

（ $f*g$ ：convolution of function f and g）

$$L[FG] = f*g$$

$$F(s) = \int_0^\infty e^{-st}f(t)dt$$

$$G(s) = \int_0^\infty e^{-st}g(t)dt$$

$$f*g = \int_0^t f(t-\tau)g(\tau)d\tau \qquad (t-\tau=\tau',\ \tau=t-\tau',\ d\tau=-d\tau')$$

$$= \int_t^0 f(\tau')g(t-\tau')(-d\tau')$$

$$= \int_0^t f(\tau)g(t-\tau)d\tau$$

pf： $L^{-1}[FG] = f*g$

$$FG = L[f*g]$$

$$F(s)G(s) = \int_0^\infty e^{-sx}f(x)dx\cdot\int_0^\infty e^{-sy}g(y)dy$$

$$= \int_0^\infty\int_0^\infty e^{-s(x+y)}f(x)g(y)dxdy$$

$$\overset{x+y=t}{\underset{y=\tau}{=}} \int_0^\infty\int_0^t e^{-st}f(t-\tau)g(\tau)d\tau dt \qquad \left(\begin{matrix} x\ge 0 & x=t-\tau\ge 0 \\ y\ge 0 & y=\tau\ge 0\end{matrix}\right)$$

$$= \int_0^\infty e^{-st}[\int_0^t f(t-\tau)g(\tau)d\tau]dt$$

$$= L[f*g]$$

$$\therefore\ L^{-1}[FG] = f*g$$

ex： $H(s)=\dfrac{1}{(s^2+1)^2}=\underbrace{\dfrac{1}{s^2+1}}_{F}\cdot\underbrace{\dfrac{1}{s^2+1}}_{G}$

$\Rightarrow\ f(t)=\sin t$ ， $g(t)=\sin t$

$\Rightarrow\ h(t)=\sin t*\sin t$

$$=\int_0^t \sin(t-\tau)\sin\tau d\tau$$

$$=\int_0^t [\sin t\cos\tau-\cos t\sin\tau]\sin\tau d\tau$$

$$=\sin t\int_0^t \underbrace{\cos\tau\sin\tau}_{=\frac{1}{2}\sin 2\tau} d\tau-\cos t\int_0^t \underbrace{\sin^2\tau}_{=\frac{1-\cos 2\tau}{2}}\, d\tau$$

$$=\sin t(-\frac{1}{4}\cos 2\tau)\Big|_0^t-\cos t(\frac{\tau}{2}-\frac{1}{4}\sin 2\tau)\Big|_0^t$$

$$=\sin t(-\frac{1}{4}\cos 2t+\frac{1}{4})-\cos t(\frac{t}{2}-\frac{1}{4}\sin 2t)$$

$$=-\frac{1}{4}\sin t\cos 2t+\frac{1}{4}\cos t\sin 2t+\frac{1}{4}\sin t-\frac{t}{2}\cos t$$

$$=-\frac{1}{4}\sin(t-2t)+\frac{1}{4}\sin t-\frac{t}{2}\cos t$$

$$=\frac{1}{4}\sin t+\frac{1}{4}\sin t-\frac{t}{2}\cos t$$

$$=\frac{1}{2}\sin t-\frac{1}{2}t\cos t$$

---

ex： $H(s)=\dfrac{1}{s^2(s-a)}=\underbrace{\dfrac{1}{s^2}}_{F}\cdot\underbrace{\dfrac{1}{s-a}}_{G}$

$\Rightarrow\ f(t)=t$ ， $g(t)=e^{at}$

$\Rightarrow\ h(t)=t*e^{at}$

$$=\int_0^t \tau e^{a(t-\tau)}d\tau$$

$$=e^{at}\int_0^t \tau e^{-a\tau}d\tau$$

$$=e^{at}[\tau(-\frac{e^{-a\tau}}{a})\Big|_0^t-\int_0^t(-\frac{e^{-a\tau}}{a})d\tau]$$

$$=e^{at}[-\frac{t}{a}e^{-at}+\frac{1}{a}(-\frac{e^{-at}}{a}+\frac{1}{a})]$$

$$\boxed{=\frac{1}{a^2}(e^{at}-at-1)}$$

$$=\int_0^t (t-\tau)e^{a\tau}d\tau$$

$$=t\int_0^t e^{a\tau}d\tau-\int_0^t \tau e^{a\tau}d\tau$$

$$=\frac{t}{a}(e^{at}-1)-(\frac{\tau}{a}e^{a\tau}\Big|_0^t-\int_0^t\frac{1}{a}e^{a\tau}d\tau)$$

$$=\frac{t}{a}e^{at}-\frac{t}{a}-\frac{t}{a}e^{at}+\frac{1}{a^2}e^{at}-\frac{1}{a^2}$$

$$\boxed{=\frac{1}{a^2}(e^{at}-at-1)}$$

---

ex：Integral equation $y(t)=t+\int_0^t y(\tau)\sin(t-\tau)d\tau$

$\Rightarrow$ Laplace transform $\Rightarrow$ $Y(s)=\dfrac{1}{s^2}+Y(s)\dfrac{1}{s^2+1}$

$$\Rightarrow\ Y(s)[1-\frac{1}{s^2+1}]=\frac{1}{s^2}$$

$$\Rightarrow\ Y(s)=\frac{s^2+1}{s^4}=\frac{1}{s^2}+\frac{1}{s^4}$$

$$\Rightarrow\ y(t)=t+\frac{1}{6}t^3$$

---

ex： $x(t)=2+\int_0^t e^{t-\tau}x(\tau)d\tau$

（ $e^t g(t)=\int_0^t e^{t-\tau}x(\tau)d\tau\ \Rightarrow\ g(t)=\int_0^t e^{-\tau}x(\tau)d\tau$ ）

$$\Rightarrow\ X(s)=\frac{2}{s}+G(s-1)$$

$$=\frac{2}{s}+\frac{1}{s-1}L[e^{-\tau}x](s-1)$$ （ $L[e^{-\tau}x](s-1)=X(s)$ ：t-shiftinf ）

$$=\frac{2}{s}+\frac{1}{s-1}X(s)$$

$$\Rightarrow\ X(s)=\frac{2(s-1)}{s(s-2)}=\frac{a}{s}+\frac{b}{s-2}$$ （ $a=1$ ， $b=1$ ）

$$=\frac{1}{s}+\frac{1}{s-2}$$

$$\Rightarrow\ x(t)=1+e^{2t}$$

---

ex： $x''(t)+4x(t)=5e^{-t}$ ， $t\ge 0$ ， $x(0)=2$ ， $x'(0)=3$

$$\Rightarrow\ [s^2X-2s-3]+4X=\frac{5}{s+1}$$

$$\Rightarrow\ X=\frac{2s^2+5s+8}{(s+1)(s^2+4)}=\frac{4}{s^2+4}+\frac{s}{s^2+4}+\frac{1}{s+4}$$

$$\Rightarrow\ x(t)=2\sin 2t+\cos 2t+e^{-t}$$

---

ex： $\begin{cases} x''+y=-2 \\ x+y''=0 \end{cases}$ ， $x(0)=y(0)=y'(0)=x'(0)=0$

$$\Rightarrow\ \begin{cases} s^2X+Y=-2 \\ X+s^2Y=0 \end{cases}\ \Rightarrow\ \begin{cases} s^4X+s^2Y=-2s^2 \\ X+s^2Y=0 \end{cases}$$

$$\Rightarrow\ X=\frac{-2s^2}{(s^2+1)(s^2-1)}=\frac{s}{s^2+1}\cdot\frac{-2s}{s^2-1}$$

$$Y=\frac{1}{(s^2+1)(s^2-1)}=\frac{1}{s^2+1}\cdot\frac{1}{s^2-1}$$

$$\Rightarrow\ y(t)=-\sin t+\sinh t$$

---

ex：Periodic Function $f(t+p)=f(t)$

$$L[f]=\int_0^\infty e^{-st}f(t)dt$$

$$= \int_0^p e^{-st} f(t)dt + \underbrace{\int_p^{2p} e^{-st} f(t)dt}_{t = t'+p} + \underbrace{\int_{2p}^{3p} e^{-st} f(t)dt}_{t = t'+2p} + \ldots\ldots$$

$$= (1 + e^{-sp} + e^{-2sp} + \ldots\ldots)\int_0^p e^{-st} f(t)dt$$

$$\int_0^p e^{-s(t'+p)} f(t'+p)dt' = e^{-sp}\int_0^p e^{-st} f(t)dt$$

$$t = p = t' + p \;\Rightarrow\; t' = 0$$

$$t = 2p = t' + p \;\Rightarrow\; t' = p$$

---

ex： $f(t) = \frac{k}{p}t$ ， $f(t+p) = f(t)$

$$\int_0^p e^{-st} t dt = (-\frac{1}{s}e^{-st})\Big|_0^p - \int_0^\infty (-\frac{1}{s}e^{-st})dt$$

$$= -\frac{1}{s}e^{-sp}p + \frac{1}{s}(-\frac{1}{s}e^{-sp} + \frac{1}{s})$$

$$= -\frac{p}{s}e^{-sp} + \frac{1}{s^2}(1 - e^{-sp})$$

$$L[f] = \frac{\frac{k}{p}}{1 - e^{-sp}}\int_0^p e^{-st} t dt = \frac{k}{ps^2} - \frac{ke^{-sp}}{s(1 - e^{-sp})}$$
