Matrix

$$A=(a_{ij})=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

（$\leftrightarrow$：row列，$\updownarrow$：column行）

（a_{ij}：matrix element，i：row，j：column）

ex：$\begin{cases} 2x+3y=1 \\ x-y=2 \end{cases} \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

matrix addition

$A+B=C \Rightarrow a_{ij}+b_{ij}=c_{ij}$

Scalar multiplication

$kA \Rightarrow k(a_{ij})$

zero matrix

(0)，每一個元素都是0

row matrix

$(\ldots \quad \ldots)$，$m=1$

column matrix

$\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$，$n=1$

square matrix

$\begin{pmatrix} \ldots & \ldots \\ \ldots & \ldots \end{pmatrix}$，$m=n$（只有方陣才有單位矩陣）

unit matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

diagonal matrix： $\begin{cases} a_{ij} \neq 0 & i = j \\ a_{ij} = 0 & i \neq j \end{cases}$

$$\begin{pmatrix} \ddots & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

A 的相關矩陣：

Transpose（轉置） $a_{ij}^T = a_{ji}$

Symmetric（對稱） $a_{ij}^T = a_{ji} = a_{ij}$ （ $A^T = A$ ）

Antisymmtric $a_{ij}^T = a_{ji} = -a_{ij}$ （ $A^T = -A$ ）

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ex： $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$ symmetric

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ex： $\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{pmatrix}$ antisymmetric

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Adjoint（伴隨） $a_{ij}^+ = (a_{ij}^T)^* = (a_{ji})^* = a_{ji}^*$

Self-adjoint ; hermitian $a_{ij}^+ = (a_{ji}^*) = a_{ij}$

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ex： $A = \begin{pmatrix} 4 & 2+i \\ 2-i & -3 \end{pmatrix}$

$\Rightarrow A^+ = \begin{pmatrix} 4 & 2-i \\ 2+i & -3 \end{pmatrix}^* = \begin{pmatrix} 4 & 2+i \\ 2-i & -3 \end{pmatrix} = A$

$\Rightarrow$ hermitian

Matrix multiplication

$$C = AB \Rightarrow c_{ij} = \sum_k a_{ik} b_{kj}$$

---

ex： $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$ ， $B = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$

$$\Rightarrow\ AB=\begin{pmatrix}2&3\\-1&1\end{pmatrix}\begin{pmatrix}1&-2\\-3&1\end{pmatrix}=\begin{pmatrix}-7&-1\\-4&3\end{pmatrix}=C$$

（$c_{11}=\sum_k a_{1k}b_{k1}=a_{11}b_{11}+a_{12}b_{21}$）

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ex：$A=\begin{pmatrix}2&1\\-3&4\end{pmatrix}$，$B=\begin{pmatrix}-7&3&2\\5&0&1\end{pmatrix}$

$$\Rightarrow\ AB=\begin{pmatrix}-9&6&9\\41&-9&-2\end{pmatrix}$$

---

ex：$\vec{A}\cdot\vec{B}=\sum_i a_ib_i$，$A=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$，$B=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

$$\Rightarrow\ \vec{A}\cdot\vec{B}=A^TB=\begin{pmatrix}a_1&a_2&a_3\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$$

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ex：$\vec{A}^*\cdot\vec{B}=\sum_i a_i^*b_i$　　（$\vec{A}^*\cdot\vec{A}=\sum_i a_i^*a_i=\sum_i|a_i|^2$）

$$\Rightarrow\ \begin{pmatrix}a_1^*&a_2^*&a_3^*\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\sum_i a_i^*b_i$$

---

ex：$(AB)C=A(BC)$　　（$AB\neq BA$）

左 $\Rightarrow\ [(AB)C]_{ij}=\sum_k(AB)_{ik}C_{kj}=\sum_k(\sum_l A_{il}B_{lk})C_{kj}=\sum_k\sum_l A_{il}B_{lk}C_{kj}$

右 $\Rightarrow\ [A(BC)]_{ij}=\sum_k A_{ik}(BC)_{kj}=\sum_k A_{ik}(\sum_l B_{kl}C_{lj})=\sum_k\sum_j A_{ik}B_{kl}C_{lj}=\sum_l\sum_k A_{il}B_{lk}C_{kj}$

∴ Associativity（結合性）

$(AB)_{ij}=\sum_k A_{ik}B_{kj}=\sum_l A_{il}B_{lj}$
ij：free index　　　k：dummy index

---

ex：$(ABC)^T=C^TB^TA^T$

左 $\Rightarrow\ (ABC)^T_{ij}=(ABC)_{ji}=\sum_l\sum_k A_{jl}B_{lk}C_{ki}$

右 $\Rightarrow\ C^TB^TA^T=\sum_l\sum_k(C^T)_{il}(B^T)_{lk}(A^T)_{kj}=\sum_l\sum_k C_{li}B_{kl}A_{jk}$

$$=\sum_l\sum_k C_{ki}B_{lk}A_{jl}=\sum_l\sum_k A_{jl}B_{lk}C_{ki}$$

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ex：trace： $trA = \sum_i a_{ii}$

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ex： $tr(ABC) = tr(BCA) = tr(CAB)$ $\Rightarrow$ cyclic permutation（輪換）

$$pf：tr(ABC) = \sum_i (abc)_{ii} = \sum_i \sum_l \sum_k A_{il} B_{lk} C_{ki}$$
$$= \sum_i \sum_l \sum_k B_{lk} C_{ki} A_{il} = tr(BCA)$$
$$= \sum_i \sum_l \sum_k C_{ki} A_{il} B_{lk} = tr(CAB)$$

---

ex： $trA^T = trA$

$$pf：trA = \sum_i a_{ii}$$
$$\Rightarrow\ trA^T = \sum_i (A^T)_{ii} = \sum_i A_{ii} = trA$$

Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}，\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}，\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$

$$\sigma_2\sigma_3 = i\sigma_1$$

$$\sigma_3\sigma_1 = i\sigma_2$$

$$\Rightarrow\ \sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3 = -\sigma_1\sigma_2$$

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$ （kronecker delta）

$$(\sigma_1)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$(\sigma_2)^2 = (\sigma_3)^2 = I$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{(ijk)是(123)的cyclic permutation} \\ -1 & \text{(ijk)是(123)的置換奇數次} \\ 0 & \text{其他} \end{cases}$$

$$\Rightarrow \begin{cases} \varepsilon_{112} = \varepsilon_{212} = \ldots\ldots = 0 \\ \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \\ \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1 \end{cases}$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2I\delta_{ij}$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \ \Rightarrow\ (\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (\sum_i \sigma_i A_i)(\sum_j \sigma_j B_j)$$

$$= \sum_i\sum_j A_i B_j \sigma_i \sigma_j$$

$$= \sum_i\sum_j A_i B_j (I\delta_{ij} + i\varepsilon_{ijk}\sigma_k)$$

$$= I\sum_i A_i B_i + i\sum \varepsilon_{ijk} A_i B_j \sigma_k \qquad (\sum_i A_i \delta_{ij} = A_i)$$

$$= \vec{A}\cdot\vec{B}I + i\vec{\delta}(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot\vec{B} = \sum_i A_i B_i \equiv A_i B_i$$

$$(\vec{A}\times\vec{B})_i = \sum_j\sum_k \varepsilon_{ijk} A_j B_k = \varepsilon_{ijk} A_j B_k$$

$$(\vec{A}\times\vec{B})_1 = A_2B_3 - A_3B_2$$

$$\sum_j\sum_k \varepsilon_{ijk} A_j B_k = A_2B_3 - A_3B_2$$

---

ex：$e^{i\vec{\sigma}\cdot\vec{\vartheta}}$ ，$\vec{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3)$

$$\because\ e^x = 1 + x + \frac{1}{2!}x^2 + \ldots\ldots$$

$$\Rightarrow\ e^{i\vec{\sigma}\cdot\vec{\vartheta}} = I + (i\vec{\sigma}\cdot\vec{\vartheta}) + \frac{1}{2!}(i\vec{\sigma}\cdot\vec{\vartheta})^2 + \frac{1}{3!}(i\vec{\sigma}\cdot\vec{\vartheta})^3 + \ldots\ldots$$

$$(\vec{\sigma}\cdot\vec{\vartheta})^2 = (\vec{\sigma}\cdot\vec{\vartheta})(\vec{\sigma}\cdot\vec{\vartheta}) = I\vec{\vartheta}^2 + i\vec{\sigma}(\vec{\vartheta}\times\vec{\vartheta}) \qquad (\ \vec{\vartheta}\times\vec{\vartheta} = 0\ )$$

$$= I + i\vec{\sigma}\cdot\vec{\vartheta} - \frac{1}{2!}\vartheta^2 I - \frac{1}{3!}(i\vec{\sigma}\cdot\vec{\vartheta})\cdot\vartheta^2 I + \ldots\ldots \qquad (\ \vec{\vartheta} = \vartheta\hat{n}\ )$$

$$= I(1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} - \ldots\ldots) + i\vec{\sigma}\cdot\hat{n}(\vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} - \ldots\ldots)$$

$$= I\cos\vartheta + i\vec{\sigma}\cdot\hat{n}\sin\vartheta$$

---

ex：$\vec{\sigma}\times\vec{\sigma} = 2i\vec{\sigma} \neq 0$

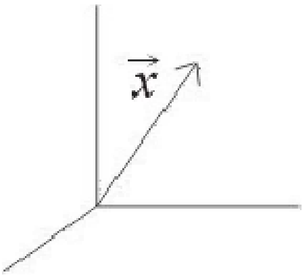
$$(\vec{\sigma}\times\vec{\sigma})_1 = \sigma_2\sigma_3 - \sigma_3\sigma_2 = 2\sigma_2\sigma_3 = 2i\sigma_1$$

---

ex：orthogonal matrix $A^T A = AA^T = I$

$$\vec{x} = (x_1, x_2, x_3)$$

$$|\vec{x}|^2 = x_1^2 + x_2^2 + x_3^2 = \sum_i x_i^2 = X^T X$$



$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\qquad X'=AX\ \text{（A：rotational matrix）}$$

$$X'^T X'=X^T X=(AX)^T(AX)=X^T\underbrace{A^T A}_{=I}X$$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\underbrace{\begin{pmatrix}\cos\vartheta & \sin\vartheta\\-\sin\vartheta & \cos\vartheta\end{pmatrix}}_{A}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\sum_k (a^T)_{ik}a_{kj}=\delta_{ij}\ \Rightarrow\ \boxed{\sum_k a_{ki}a_{kj}=\delta_{ij}}\ \text{……正交條件}$$

---

ex： $A=\begin{pmatrix}\cos\vartheta & \sin\vartheta\\-\sin\vartheta & \cos\vartheta\end{pmatrix}$

$\Rightarrow$ ① $i=1$，$j=2$

$(\cos\vartheta,\sin\vartheta)\cdot(-\sin\vartheta,\cos\vartheta)=0$ ……正交

② $i=j=1$，$\cos^2\vartheta+(-\sin\vartheta)^2=1$

---

ex：Unitary matrix

$U^+U=UU^+=I$

$$(U^+U)_{ij}=\delta_{ij}\ \Rightarrow\ \sum_k (U^+)_{ik}U_{kj}=\delta_{ij}\ \Rightarrow\ \boxed{\sum_k U^*_{ki}U_{kj}=\delta_{ij}}$$

$$(U^+U)_{ij}=\delta_{ij}\ \Rightarrow\ \sum_k U_{ik}(U^+)_{kj}=\delta_{ij}\ \Rightarrow\ \boxed{\sum_k U_{ki}U^*_{kj}=\delta_{ij}}$$

Second-order Determinant

$$D=delA=\begin{vmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

---

ex： $\begin{cases}4x_1+3x_2=12\\2x_1+5x_2=-8\end{cases}\Rightarrow\begin{pmatrix}4 & 3\\2 & 5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}12\\-8\end{pmatrix}$

$$D=\begin{vmatrix}4 & 3\\2 & 5\end{vmatrix}=14$$

$$x_1=\frac{D_1}{D}\text{，}\ D_1=\begin{vmatrix}12 & 3\\-8 & 5\end{vmatrix}=84\text{，}\ \therefore\ x_1=6$$

$$x_2=\frac{D_2}{D}\text{，}\ D_2=\begin{vmatrix}4 & 12\\2 & -8\end{vmatrix}=-56\text{，}\ \therefore\ x_2=-4$$

Third-order determinant

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

$$x_1 = \frac{D_1}{D}\text{ ， } x_2 = \frac{D_2}{D}\text{ ， } x_3 = \frac{D_3}{D}$$

N-order determinant

$$D = delA = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$D = \sum_{\sigma} (-1)^{\sigma} a_{1\sigma(1)} \ldots\ldots a_{n\sigma(n)}$　　　　σ ：permutation 相鄰交換的次數

---

ex： $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$

$D = a_{j1}C_{j1} + a_{j2}C_{j2} + a_{j3}C_{j3} + \ldots\ldots + a_{jn}C_{jn}$　　　C_{j1} ：cofactor

$C_{jk} = (-1)^{j+k} \underbrace{M_{jk}}_{minor}$

M_{jk} ：原行列式中除去第 j 個 row，第 k 個 column 所形成的子行列式

---

ex： $D = delA = |A| = \begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 4 & 1 & -1 & -1 \\ 1 & 2 & 3 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & -1 \\ 2 & 3 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 & 3 \\ 1 & -1 & -1 \\ 2 & 3 & 0 \end{vmatrix} + 4\begin{vmatrix} -1 & 2 & 3 \\ 2 & 0 & 2 \\ 2 & 3 & 0 \end{vmatrix} - \begin{vmatrix} -1 & 2 & 3 \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$= 16 - 2\cdot 8 + 4\cdot 32 - 0 = 128$

---

ex： $del(AB) = delA \cdot delB$　　　（ $del(A+B) \neq delA + delB$ ）

$$
\begin{aligned}
del(AB) &= del(\sum_k A_{ik}B_{kj}) \\
&= \sum_\sigma (-1)^\sigma (\sum_{k_1} A_{1k_1}B_{k_1\sigma(1)})(\sum_{k_2} A_{2k_2}B_{k_2\sigma(2)})\ldots\ldots(\sum_{k_n} A_{nk_n}B_{k_n\sigma(n)}) \\
&= \sum_\sigma \sum_{k_1}\sum_{k_2}\ldots\sum_{k_n} (-1)^\sigma A_{1k_1}A_{2k_2}\ldots A_{nk_n}B_{k_1\sigma(1)}B_{k_2\sigma(2)}\ldots B_{k_n\sigma(n)} \\
&= \sum_{k_1}\sum_{k_2}\ldots\sum_{k_n} A_{1k_1}A_{2k_2}\ldots A_{nk_n}[\sum_\sigma (-1)^\sigma B_{k_1\sigma(1)}B_{k_2\sigma(2)}\ldots B_{k_n\sigma(n)}] \\
&= (delA)(delB)
\end{aligned}
$$

Inverse Matrix （行列式不得爲零）

$AB = BA = I \quad \Rightarrow \quad B = A^{-1}$

$(A^{-1})_{ij} = \dfrac{cofactor\ \ of\ \ A_{ji}}{delA}$

$delA \neq 0$

$delA = 0$ 　則稱 A 爲singular

pf： $C_{ij} =$ cofactor of A_{ij}

$$
\begin{aligned}
[A\cdot C^T]_{ij} &= (A\text{的第}i\text{個}row)\cdot(C^T\text{的第j個column}) \\
&= \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix}\begin{pmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{pmatrix} \\
&= a_{i1}C_{j1} + a_{i2}C_{j2} + \ldots + a_{in}C_{jn} \\
&= \begin{cases} delA & i = j \\ 0 & i \neq j \end{cases}
\end{aligned}
$$

ex：$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & 4 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 0$

若 $a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = a_{21}c_{21} + a_{22}c_{12} + a_{23}c_{13}$ 則此行列式爲零

ex：$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$(ABC)(ABC)^{-1} = ABCC^{-1}B^{-1}A^{-1} = I$

$(ABC)^{-1}(ABC) = C^{-1}B^{-1}A^{-1}ABC = I$

ex：$\begin{cases} x_1 + 3x_2 + x_3 = -2 \\ 2x_1 + x_2 + x_3 = -5 \\ x_1 + 2x_2 + 3x_3 = 6 \end{cases}$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 6 \end{pmatrix}$$

$\Rightarrow AX = B$

$\Rightarrow X = A^{-1}B$

$\Rightarrow x_1 = 1$，$x_2 = -2$，$x_3 = 3$

應用數學

Eigenvalue Problem

固有值、本徵值

$AX = \lambda X$

λ ：eigenvalue

X ：eigenvector

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \lambda \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

---

ex： $A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \Rightarrow AX = \lambda X$

$$\Rightarrow \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 6 = 0 \Rightarrow \lambda = -1, -6$$

$$\lambda = -1 \Rightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases} \Rightarrow X^{(1)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda = -6 \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases} \Rightarrow X^{(2)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

---

ex： $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^3+\lambda^2-21\lambda-45=0$$
$$\Rightarrow (\lambda-5)(\lambda+3)^2=0$$

---

ex：stretching of an elastic membrane

circle $x_1^2+x_2^2=1$

$(x_1,x_2)\to(y_1,y_2)$

$$\begin{cases} y_1=5x_1+3x_2 \\ y_2=3x_1+5x_2 \end{cases} \Rightarrow Y=AX\text{，}A=\begin{pmatrix}5&3\\3&5\end{pmatrix}\text{，}AX=\lambda X$$

$$\Rightarrow \begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-10\lambda+16=0 \Rightarrow \lambda=8,2$$

$$\lambda=8 \Rightarrow X^{(1)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \Rightarrow \frac{Z_1^2}{8^2}+\frac{Z_2^2}{2^2}=1$$

$$\lambda=2 \Rightarrow X^{(2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

應用數學

Similarity transformation

$$AX=\lambda X \Rightarrow R^{-1}AX=\lambda R^{-1}X \qquad A'=R^{-1}AR\text{，}Y=R^{-1}X$$

$$\Rightarrow \underbrace{R^{-1}AR}_{A'}\,\underbrace{R^{-1}X}_{Y}=\lambda\,\underbrace{R^{-1}X}_{Y}$$

$$\Rightarrow A'Y=\lambda Y$$

A 和 A' 有相同的eigenvalue

---

ex：$A=\begin{pmatrix}5&4\\1&2\end{pmatrix} \Rightarrow |A-\lambda I|=\begin{vmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{vmatrix}$

$$\Rightarrow \lambda^2-7\lambda+6=0$$
$$\Rightarrow \lambda=1,6$$

$$\lambda=1 \Rightarrow \begin{pmatrix}5-1 & 4\\ 1 & 2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0 \Rightarrow x+y=0 \Rightarrow X^{(1)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\lambda=6 \Rightarrow \begin{pmatrix}-1 & 4\\ 1 & -4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0 \Rightarrow x-4y=0 \Rightarrow X^{(1)}=\frac{1}{\sqrt{17}}\begin{pmatrix}4\\1\end{pmatrix}$$

$$R=\frac{1}{\sqrt{34}}\begin{pmatrix}1&4\\-1&1\end{pmatrix} \Rightarrow R^{-1}=\frac{\sqrt{34}}{5}\begin{pmatrix}1&-4\\1&1\end{pmatrix}$$

$$A'=R^{-1}AR=\frac{1}{5}\begin{pmatrix}1&-4\\1&1\end{pmatrix}\begin{pmatrix}5&4\\1&2\end{pmatrix}\begin{pmatrix}1&4\\-1&1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}5&0\\0&30\end{pmatrix}=\begin{pmatrix}1&0\\0&6\end{pmatrix}$$

---

ex：$A=\begin{pmatrix}5&4\\1&2\end{pmatrix}$，$A^{50}=?$

$$\Rightarrow f(A)=A^{50}=R(\underbrace{R^{-1}AR}_{A'})(\underbrace{R^{-1}AR}_{A'})\ldots\ldots(\underbrace{R^{-1}AR}_{A'})R^{-1}=R(A')^{50}R^{-1}$$

$$A'=\begin{pmatrix}1&0\\0&6\end{pmatrix},\ R=\frac{1}{\sqrt{34}}\begin{pmatrix}1&4\\-1&1\end{pmatrix},\ R^{-1}=\frac{\sqrt{34}}{5}\begin{pmatrix}1&-4\\1&1\end{pmatrix}$$

$$A^{50}=\frac{1}{5}\begin{pmatrix}1&4\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&6^{50}\end{pmatrix}\begin{pmatrix}1&-4\\1&1\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}1&4\\-1&1\end{pmatrix}\begin{pmatrix}1&-4\\6^{50}&6^{50}\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}1+4\times6^{50}&-4+4\times6^{50}\\-1+6^{50}&4+6^{50}\end{pmatrix}$$

Invariant（不變量）

$\Rightarrow$ TrA，$delA$

$TrA'=TrR^{-1}AR=TrARR^{-1}=TrA$ $\qquad Tr(abc)=Tr(bca)=Tr(cab)$

$TrA=$ eigenvalue的總和

$delA'=del(R^{-1}AR)=(delR^{-1})(delA)(delR)=delA$

$del(R^{-1}R)=del(I)=1=(delR^{-1})(delR)$ $\Rightarrow\ delR^{-1}=(delR)^{-1}$

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ex：$H=\begin{pmatrix}2&-1&-3\\-1&1&2\\-3&2&3\end{pmatrix}$，(a) $\sum_i\lambda_i=?$ (b) $\sum_i\lambda_i^2=?$

$\Rightarrow$ (a) $\sum_i\lambda_i=2+1+3=6$

(b) $H^2=\begin{pmatrix}2&-1&-3\\-1&1&2\\-3&2&3\end{pmatrix}\begin{pmatrix}2&-1&-3\\-1&1&2\\-3&2&3\end{pmatrix}=\begin{pmatrix}4+1+9&&\\&1+1+4&\\&&9+4+9\end{pmatrix}$

$\sum_i\lambda_i^2=(4+1+9)+(1+1+4)+(9+4+9)=42$

$A^2=\begin{pmatrix}\lambda_1&&\\&\lambda_2&\\&&\lambda_3\end{pmatrix}\begin{pmatrix}\lambda_1&&\\&\lambda_2&\\&&\lambda_3\end{pmatrix}=\begin{pmatrix}\lambda_1^2&&\\&\lambda_2^2&\\&&\lambda_3^2\end{pmatrix}$

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ex：eigenvalues（有重根）（degenerate）

degenerate ：一個 $\lambda\leftrightarrow$ 多個eigenvector

nondegenerate ：一個 $\lambda\leftrightarrow$ 一個eigenvector

$$A=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\Rightarrow |A-\lambda I|=\begin{vmatrix}-\lambda&0&0&1\\0&-\lambda&1&0\\0&1&-\lambda&0\\1&0&0&-\lambda\end{vmatrix}$$

$$=-\lambda\begin{vmatrix}-\lambda&1&0\\1&-\lambda&0\\0&0&-\lambda\end{vmatrix}-\begin{vmatrix}0&0&1\\-\lambda&1&0\\1&-\lambda&0\end{vmatrix}$$

$$\Rightarrow (\lambda^2-1)^2=0 \quad \Rightarrow \quad \lambda=1,1,-1,-1$$

$$\Rightarrow \lambda=1，X^{(1)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}，X^{(2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$X^{(2)}$ 的決定是要與 $X^{(1)}$ 正交比較好 $\Rightarrow X^{(1)T}\cdot X^{(2)}=0$

$$\Rightarrow \lambda=-1，X^{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}，X^{(4)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

應用數學

Conic Section

Quadratic form（二次項）

$$Q=X^TAX=\sum_{j=1}^{3}\sum_{k=1}^{3}a_{jk}X_jX_k$$

$$=a_{11}X_1^2+a_{22}X_2^2+a_{33}X_3^2+2a_{12}X_1X_2+2a_{23}X_2X_3+2a_{31}X_3X_1$$

Transformation to principal axes

$$Q=X^TAX=\underbrace{X^TR}_{Y^T}\underbrace{R^{-1}AR}_{A'}\underbrace{R^{-1}X}_{Y}=Y^TA'Y=\begin{pmatrix}y_1&y_2\end{pmatrix}\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$(R^{-1})^T=R \Rightarrow R^{-1}=R^T$

$Y=R^{-1}X \Rightarrow Y^T=X^T(R^{-1})^T=X^TR$

$X^TX=$ 不變量

$X\to RX=X'$ $\qquad X^T\to X^TR^T=X'^T$

$\Rightarrow X'^TX'=X^T\underbrace{R^TR}_{I}X=X^TX$

$R^{-1}=R^T$ (symmetric) $\overset{右乘R}{\Rightarrow} R^{-1}R=R^TR \Rightarrow I=R^TR$

ex： $Q=17X_1^2-30X_1X_2+17X_2^2=128$

$$= X^T AX$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 17 & -15 \\ -15 & 17 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 17x_1 - 15x_2 \\ -15x_1 + 17x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 17 & -15 \\ -15 & 17 \end{pmatrix} \Rightarrow \begin{vmatrix} 17 & -15 \\ -15 & 17 \end{vmatrix} = 0 \Rightarrow \lambda = 2, 32$$

$$\lambda = 2 \quad \Rightarrow \quad X^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 32 \quad \Rightarrow \quad X^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\because$ $\boxed{Q = \lambda_1 Y_1^2 + \lambda_2 Y_2^2} = 2Y_1^2 + 32Y_2^2 = 128$ ......爲一橢圓

$$Y = R^{-1}X \quad \Rightarrow \quad X = RY = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\therefore \quad X_1 = \frac{Y_1}{\sqrt{2}} - \frac{Y_2}{\sqrt{2}}$ ， $X_2 = \frac{Y_1}{\sqrt{2}} + \frac{Y_2}{\sqrt{2}}$

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ex： $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2 - xy + y^2)}dxdy \Rightarrow Q = x^2 - xy + y^2 = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = X^T AX$

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2 - \frac{1}{4} = 0 \Rightarrow \lambda = \frac{3}{2}, \frac{1}{2}$$

$$Q = \frac{3}{2}S_1^2 + \frac{1}{2}S_2^2$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\frac{3}{2}S_1^2 + \frac{1}{2}S_2^2)} JdS_1 dS_2 = \sqrt{\frac{2\pi}{3}} \cdot \sqrt{2\pi} = \frac{2\pi}{\sqrt{3}}$$

$$\lambda = \frac{3}{2} \quad \Rightarrow \quad X^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = \frac{1}{2} \quad \Rightarrow \quad X^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$R^{-1}X=Y \Rightarrow X=RY=\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}S_1\\ S_2\end{pmatrix}$$

$$J=\left\|\begin{matrix}\frac{\partial X}{\partial S_1} & \frac{\partial X}{\partial S_2}\\ \frac{\partial Y}{\partial S_1} & \frac{\partial Y}{\partial S_2}\end{matrix}\right\|=\left\|\begin{matrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{matrix}\right\|=1$$

---

ex： $\begin{cases}\dot{x}_1=-4x_1+x_2+x_3\\ \dot{x}_2=x_1+5x_2-x_3\\ \dot{x}_3=x_2-3x_3\end{cases}$

$$\Rightarrow \dot{X}=AX$$

$$\therefore A=\begin{pmatrix}-4 & 1 & 1\\ 1 & 5 & -1\\ 0 & 1 & 3\end{pmatrix}，X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$\Rightarrow \underbrace{R^{-1}\dot{X}}_{\dot{Y}}=\underbrace{R^{-1}AR}_{A'}\underbrace{R^{-1}X}_{Y} \Rightarrow \begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix}=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

$$\Rightarrow |A-\lambda I|=\begin{vmatrix}-4-\lambda & 1 & 1\\ 1 & 5-\lambda & -1\\ 0 & 1 & 3-\lambda\end{vmatrix}=0 \Rightarrow \lambda=-3,-4,5$$

$$\lambda=-3 \Rightarrow X^{(1)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$$

$$\lambda=-4 \Rightarrow X^{(2)}=\frac{1}{\sqrt{102}}\begin{pmatrix}10\\ -1\\ 1\end{pmatrix}$$

$$\lambda=5 \Rightarrow X^{(3)}=\frac{1}{\sqrt{66}}\begin{pmatrix}1\\ 8\\ 1\end{pmatrix}$$

$$R=\frac{1}{\sqrt{2\times102\times66}}\begin{pmatrix}1 & 10 & 1\\ 0 & -1 & 8\\ 1 & 1 & 1\end{pmatrix} \Rightarrow R^{-1}=\frac{1}{72}\begin{pmatrix}-9 & -9 & 81\\ 8 & 0 & -8\\ 1 & 9 & -1\end{pmatrix}$$

$$Y=R^{-1}X \Rightarrow X=RY$$

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix}=\begin{pmatrix}-3&0&0\\0&-4&0\\0&0&5\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\Rightarrow\begin{cases}\dot{y}_1=-3y_1\\ \dot{y}_2=-4y_2\\ \dot{y}_3=5y_3\end{cases}\Rightarrow\begin{cases}y_1=c_1e^{-3t}\\ y_2=c_2e^{-4t}\\ y_3=c_3e^{5t}\end{cases}$$

$$\therefore\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1&10&1\\0&-1&8\\1&1&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

$$\begin{cases}x_1=c_1e^{-3t}+10c_2e^{-4t}+c_3e^{5t}\\ x_2=-c_2e^{-4t}+8c_3e^{5t}\\ x_3=c_1e^{-3t}+c_2e^{-4t}+c_3e^{5t}\end{cases}$$

pf：$\int_{-1}^{1}[P_n(x)]^2\,dx=\dfrac{2}{2n+1}$

$\Rightarrow$ 由生成函數 $(1-2xu+u^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)u^n$

$$\Rightarrow\int_{-1}^{1}(1-2xu+u^2)^{-1}dx=\sum_m\sum_n\left(\int_{-1}^{1}P_m(x)P_n(x)\right)u^{m+n}$$

$$=\sum_n\int_{-1}^{1}[P_n(x)]^2u^{2n}\qquad (\ y=1-2xu+u^2\ )$$

$$\Rightarrow\int_{(1+u)^2}^{(1-u)^2}\left(\frac{1}{-2u}\right)\frac{dy}{y}=\frac{1}{2u}\ln y\Big|_{(1-u)^2}^{(1+u)^2}=\frac{1}{2u}\ln\frac{(1+u)^2}{(1-u)^2}=\frac{1}{u}\ln\frac{1+u}{1-u}$$

$$\ln(1+u)=u-\frac{u^2}{2}+\frac{u^3}{3}-\ldots\ldots$$

$$\ln(1-u)=-u-\frac{u^2}{2}-\frac{u^3}{3}-\ldots\ldots$$

$$\Rightarrow\frac{1}{u}\ln\frac{1+u}{1-u}=\frac{1}{u}[\ln(1+u)-\ln(1-u)]$$

$$=\frac{1}{u}[2u+\frac{2}{3}u^3+\frac{2}{5}u^5+\ldots\ldots]$$

$$=2[1+\frac{1}{3}u^2+\frac{1}{5}u^4+\ldots\ldots]$$

$$=2\sum_{n=0}^{\infty}\frac{u^{2n}}{2n+1}$$

$$\Rightarrow 2\sum_{n=0}^{\infty}\frac{u^{2n}}{2n+1}=\sum_{n=0}^{\infty}\int_{-1}^{1}[P_n(x)]^2\,dx\ u^{2n}$$

$\Rightarrow\int_{-1}^{1}[P_n(x)]^2\,dx=\dfrac{2}{2n+1}$ ……得證

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2^{m+n}m!n!}\int_{-1}^{1}[\frac{d^m}{dx^m}(x^2-1)^m][\frac{d^n}{dx^n}(x^2-1)^n]dx \text{……利用部分積分}$$

$$\overset{m>n}{=}(-1)^m\frac{1}{2^{m+n}m!n!}\int_{-1}^{1}(x^2-1)^m\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n dx$$