Vector Analysis

$$\vec{a}=(a_1,a_2,a_3)=a_1e_1+a_2e_2+a_3e_3=\sum_{i=1}^{3}a_ie_i=a_ie_i$$

（Einstein convention）

$e_1=(1,0,0)$，$e_2=(0,1,0)$，$e_3=(0,0,1)$……單位向量

---

Scalar product（doc，inner）

define：$e_i\cdot e_j=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$（直角座標系才適用）

（$\delta_{ij}$：kronecker delta）

$$\vec{A}\cdot\vec{B}=(\sum_i a_ie_i)\cdot(\sum_j b_je_j)=\sum_i\sum_j a_ib_je_i\cdot e_j=\sum_i\sum_j a_ib_j\delta_{ij}=\sum_i a_ib_i=a_1b_1+a_2b_2+a_3b_3$$

$$\sum_{i=1}^{3}a_ib_i=\sum_{j=1}^{3}a_jb_j$$ （$i$、$j$：dummy index）

---

Cross product（exterior）

define：$e_i\times e_j=\sum_k\varepsilon_{ijk}e_k=\varepsilon_{ijk}e_k$

Levi-Civita Symbol：$\varepsilon_{ijk}=\begin{cases}1 & \varepsilon_{123}=\varepsilon_{231}=\varepsilon_{312}\\ -1 & \varepsilon_{213}=\varepsilon_{132}=\varepsilon_{321}\\ 0 & \text{其他：}\varepsilon_{112}=\varepsilon_{223}\end{cases}$

$e_1\times e_2=e_3=-e_2\times e_1$

$e_2\times e_3=e_1=-e_3\times e_2$

$e_3\times e_1=e_2=-e_1\times e_3$

$e_1\times e_1=e_2\times e_2=e_3\times e_3=0$

$$\begin{aligned}\vec{A}\times\vec{B}&=(\sum_i a_ie_i)\times(\sum_j b_je_j)=\sum_i\sum_j a_ib_je_i\times e_j=\sum_i\sum_j a_ib_j(\sum_k\varepsilon_{ijk}e_k)\\&=\sum_i\sum_j\sum_k\varepsilon_{ijk}a_ib_je_k\\&=\begin{vmatrix}e_1 & e_2 & e_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\&=(a_2b_3-a_3b_2)e_1+(a_3b_1-a_1b_3)e_2+(a_1b_2-a_2b_1)e_3\end{aligned}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} e_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_3$$

---

ex： $\vec{A} = (1,0,-1)$ ， $\vec{B} = (2,1,3)$

$$\Rightarrow \ \vec{A} \times \vec{B} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{vmatrix} = (1,-5,1)$$

---

ex： $\vec{C} = \vec{A} \times \vec{B} \ \Rightarrow \ \left|\vec{A} \times \vec{B}\right| = AB\sin\vartheta$

$$\Rightarrow \ \left|\vec{A} \times \vec{B}\right|^2 + (\vec{A} \cdot \vec{B})^2 = A^2 B^2 \qquad ( A \equiv \left|\vec{A}\right| )$$

$$\vec{C} \cdot \vec{A} = \vec{A} \cdot \vec{C} = \vec{A} \cdot (\vec{A} \times \vec{B})$$

$$= (a_1 e_1 + a_2 e_2 + a_3 e_3) \cdot ( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} e_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_3 )$$

$$= a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2 B_3 - A_3 B_2)\hat{i} + (A_3 B_1 - A_1 B_3)\hat{j} + (A_1 B_2 - A_2 B_1)\hat{k}$$

$$= C_1 \hat{i} + C_2 j + C_3 \hat{k}$$

$$\left|\vec{A} \times \vec{B}\right| = \sqrt{C_1^2 + C_2^2 + C_3^2}$$

---

Triple scalar product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \sum_i a_i (\vec{b} \times \vec{c})_i = \sum_i a_i (\sum_j \sum_k \varepsilon_{ijk} b_j c_k) = \sum_i \sum_j \sum_k \varepsilon_{ijk} a_i b_j c_k$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

---

Triple vector product

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$
$$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$$

$pf$：$[\vec{a}\times(\vec{b}\times\vec{c})]_1=a_2(\vec{b}\times\vec{c})_3-a_3(\vec{b}\times\vec{c})_2$

$$=a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)$$
$$=(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1$$
$$=(\vec{a}\cdot\vec{c})b_1-(\vec{a}\cdot\vec{b})c_1$$
$$\therefore\ \vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

$$\varepsilon_{ijk}\varepsilon_{lmk}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$
$$[(\vec{a}\times\vec{b})\times\vec{c}]_i=\varepsilon_{ijk}(\vec{a}\times\vec{b})_jc_k=\varepsilon_{ijk}\varepsilon_{jlm}a_lb_mc_k$$
$$=-\varepsilon_{ikj}\varepsilon_{lmj}a_lb_mc_k=-(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl})a_lb_mc_k$$
$$=-(a_ib_mc_m-a_lb_ic_l)=-(\vec{b}\cdot\vec{c})a_i+(\vec{a}\cdot\vec{c})b_i$$

---

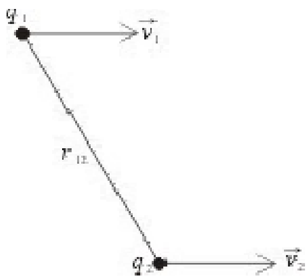
ex：電荷 $q$，速度 $v$，$\vec{B}_1=\frac{\mu_0}{4\pi}q_1\frac{\vec{v}_1\times\vec{r}_{12}}{r_{12}^2}$，$r_{12}=r$，$\vec{r}_{12}=-\vec{r}_{21}$

$$\vec{F}_2=q_2(\vec{v}_2\times\vec{B}_1)=\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^2}\vec{v}_2\times(\vec{v}_1\times\vec{r}_{12})$$
$$\vec{v}_2\times(\vec{v}_1\times\vec{r}_{12})=\vec{v}_1(\vec{v}_2\cdot\vec{r}_{12})-\vec{r}_{12}(\vec{v}_2\cdot\vec{v}_1)$$

$$\vec{F}_1=q_1(\vec{v}_1\times\vec{B}_2)=\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^2}\vec{v}_1\times(\vec{v}_2\times\vec{r}_{21})=-\vec{F}_2$$
$$\vec{v}_1\times(\vec{v}_2\times\vec{r}_{21})=\vec{v}_2(\vec{v}_1\cdot\vec{r}_{21})-\vec{r}_{21}(\vec{v}_1\cdot\vec{v}_2)$$

$\therefore$ 牛頓第三定律成立

$\Rightarrow$ 牛頓三定律不成立，除非 $\vec{v}_1 // \vec{v}_2$




---

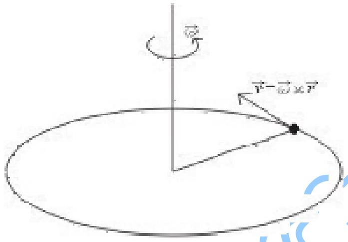
ex：$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

左式 $=\vec{c}[\vec{d}\times(\vec{a}\times\vec{b})]$

$$=\vec{c}[\vec{a}(\vec{b}\cdot\vec{d})-\vec{b}(\vec{a}\cdot\vec{d})]$$
$$=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})=\text{右式}$$

ex：rotational motion

$$\vec{E}=\frac{1}{2}m(\vec{\omega}\times\vec{r})^2$$
$$=\frac{1}{2}m[\omega^2r^2-(\vec{\omega}\cdot\vec{r})^2]$$
$$=\frac{1}{2}m(\sum_i\sum_j\omega_i\omega_i r_j r_j-\sum_i\sum_j\omega_i r_i\omega_j r_j) \quad =\frac{1}{2}m(\omega_i\omega_j r^2-\omega_i r_i\omega_j r_j)$$
$$=\frac{1}{2}m\sum_{i,j}\underbrace{(\delta_{ij}r^2-r_i r_j)}_{=I_{ij}}\omega_i\omega_j \quad =\frac{1}{2}m\underbrace{(r^2\delta_{ij}-r_i r_j)}_{=I_{ij}}\omega_i\omega_j$$
$$=\frac{1}{2}\sum_i\sum_j[m(\delta_{ij}r^2-r_i r_j)\omega_i\omega_j]$$




ex：$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\cdot(\vec{b}\times\vec{d})]\vec{c}-[\vec{a}\cdot(\vec{b}\times\vec{c})]\vec{d}$

左式：$=\vec{c}[(\vec{a}\times\vec{b})\cdot\vec{d}]-\vec{d}[(\vec{a}\times\vec{b})\cdot\vec{c}]$

$$=[\vec{a}\cdot(\vec{b}\times\vec{d})]\vec{c}-[\vec{a}\cdot(\vec{b}\times\vec{c})]\vec{d}$$

應用數學

Gradient（梯度）

scalar field

$$d\phi(x,y,z)=\phi(x+dx,y+dy,z+dz)-\phi(x,y,z)$$
$$=\phi(x,y,z)+(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz+\underbrace{......}_{=0})-\phi(x,y,z)$$
$$=(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z})\cdot(dx,dy,dz)$$
$$=\nabla\phi\cdot d\vec{r}$$
$$\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$$

$\nabla\phi$：$\phi$的梯度，Geometrical properties of $\nabla\phi$

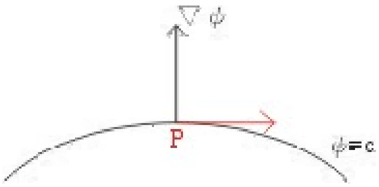
(1) $\nabla\phi$ 垂直於$\phi(x,y,z)$＝常數

(2) $\nabla\phi$ 沿著$\phi$在空間變化率最大的方向

$pf$：(1) $d\phi=0 \Rightarrow$ P 沿著移動$\phi=c$

$$\Rightarrow\nabla\phi\cdot d\vec{r}=0$$
$$\Rightarrow\nabla\phi\perp d\vec{r}$$

$\therefore$ 在 P 處 $\nabla\phi$ 垂直於 $\phi = c$

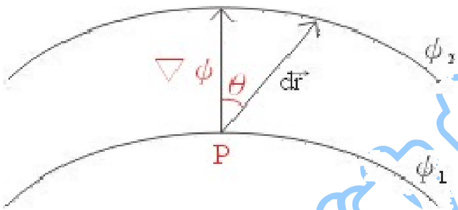



(2) $d\phi = \phi_1 - \phi_2 = \nabla\phi \cdot d\vec{r} = |\nabla\phi||d\vec{r}|\cos\vartheta$

$\vartheta = 0$ ， $\cos 0 = 1$ ， max

$\therefore$ $|d\vec{r}|$ min

$\nabla\phi$ 沿著 $\phi$ 空間變化率最大的方向




---

ex： $\phi = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ ，(a) $\nabla\phi$ 在 $(1,2,3)$

(b) $|\nabla\phi|$ 在 $(1,2,3)$

(a) $\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$

$$\frac{\partial\phi}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}} \cdot 2x = -x(x^2 + y^2 + z^2)^{-\frac{3}{2}} = -\frac{1}{(14)^{\frac{3}{2}}}$$

$$\therefore \nabla\phi = (-\frac{1}{14^{\frac{3}{2}}}, -\frac{2}{14^{\frac{3}{2}}}, -\frac{3}{14^{\frac{3}{2}}})$$

(b) $|\nabla\phi| = \frac{14^{\frac{1}{2}}}{14^{\frac{3}{2}}} = \frac{1}{14}$

---

ex：球 $x^2 + y^2 + z^2 = 1$ ，求在 P 點 $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ 之法向量

Let $\phi = x^2 + y^2 + z^2$

$$\nabla\phi = (2x, 2y, 2z) = (\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \overset{\text{取單位向量}}{\Rightarrow} (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

If $(x - \frac{1}{\sqrt{3}}, y - \frac{1}{\sqrt{3}}, z - \frac{1}{\sqrt{3}}) \cdot (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = 0$

$\Rightarrow$ $x + y + z = \sqrt{3}$ ……P 點的切平面

ex：electric potential $V=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ ，$\vec{E}=-\nabla V$ ，$\vec{r}=(x,y,z)$

$$\nabla r=\nabla\sqrt{x^2+y^2+z^2}=(\frac{\partial r}{\partial x},\frac{\partial r}{\partial y},\frac{\partial r}{\partial z})=(\frac{x}{r},\frac{y}{r},\frac{z}{r})=\frac{1}{r}(x,y,z)=\frac{\vec{r}}{r}=\hat{r}$$

$$\nabla V=\frac{q}{4\pi\varepsilon_0}\nabla(\frac{1}{r})=\frac{q}{4\pi\varepsilon_0}(-\frac{1}{r^2})\hat{r}=-\vec{E}$$

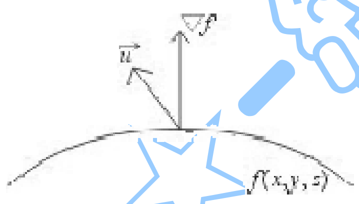
ex： $f=2x^2+3y^2+z^2$ 在 $(2,1,3)$ 沿著 $\vec{u}=(1,0,-2)$ 的方向導數（directional derivative）

$$\nabla f|_{(2,1,3)}=(4x,6y,2z)|_{(2,1,3)}$$

$$\nabla f|_{(2,1,3)}\cdot\hat{u}=(8,6,6)\cdot(\frac{1}{\sqrt{5}},0,-\frac{2}{\sqrt{5}})$$

$$=-\frac{4}{\sqrt{5}}=|\nabla f|\cos\vartheta$$

$$\therefore\ u=-\frac{4}{\sqrt{5}}$$




ex：a vector field $\vec{F}(x,y,z)$ ，$\frac{d\vec{F}}{dt}=\frac{\partial\vec{F}}{\partial t}+(\vec{v}\cdot\nabla)\vec{F}$

$$\frac{d\vec{F}}{dt}=\frac{\partial\vec{F}}{\partial t}+\frac{dx}{dt}\frac{\partial}{\partial x}\vec{F}+\frac{dy}{dt}\frac{\partial}{\partial y}\vec{F}+\frac{dz}{dt}\frac{\partial}{\partial z}\vec{F}$$

$$=\frac{\partial\vec{F}}{\partial t}+(\vec{v}\cdot\nabla)\vec{F}$$



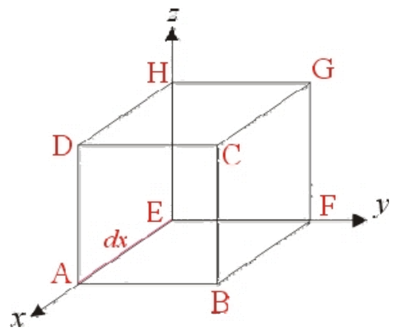
Divergence 散度

$$\vec{v}=(v_x,v_y,v_z)$$

$$\nabla\cdot\vec{v}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

$\vec{J}=\rho\vec{v}$ （ $\vec{J}$ ：current density；flux 通量）

$[\vec{J}]=\frac{\#}{m^2\cdot\sec}$ （#：指的是質量、電量、能量……）




rate of flow out $\left|_{ABCD} = \rho v_x\right|_{x=dx} dydz = [\rho v_x|_{x=0} + \frac{\partial(\rho v_x)}{\partial x}\Big|_{x=0} dx + ......]dydz$ ……(1)

rate of flow out $\left|_{EFGH} = \rho v_x\right|_{x=0} \underbrace{(-dydz)}_{-x方向}$ ……(2)

$\underbrace{(1)+(2)}_{x方向和-x方向之和} \Rightarrow \frac{\partial(\rho v_x)}{\partial x} dxdydz$

同理，y 和 z 方向亦同

net flow out (per unit time) $= [\nabla\cdot(\rho\vec{v})]dxdydz$

$\nabla\cdot(\rho\vec{v}) = \lim_{d\tau\to 0}\frac{單位時間淨流出的量}{d\tau}$

ex： $\vec{r} = (x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}$

$\nabla\cdot\vec{r} = 1+1+1 = 3$

ex： $\nabla\cdot(f\vec{v}) = (\nabla f)\vec{v} + f(\nabla\cdot\vec{v})$

$$\nabla\cdot(f\vec{v}) = \frac{\partial}{\partial x}(fv_x) + \frac{\partial}{\partial y}(fv_y) + \frac{\partial}{\partial z}(fv_z)$$

$$= \frac{\partial f}{\partial x}v_x + f\frac{\partial v_x}{\partial x} + \frac{\partial f}{\partial y}v_y + f\frac{\partial v_y}{\partial y} + \frac{\partial f}{\partial z}v_z + f\frac{\partial v_z}{\partial z}$$

$$= (\nabla f)\vec{v} + f(\nabla\cdot\vec{v})$$

ex： $r = |\vec{r}|$ ， $\hat{r} = \frac{\vec{r}}{r}$

$$\nabla\cdot(r^n\hat{r}) = \nabla\cdot(r^{n-1}\vec{r}) = (\nabla r^{n-1})\cdot\vec{r} + r^{n-1}(\nabla\cdot\vec{r})$$

$$= (n-1)r^{n-2}\underbrace{\hat{r}\cdot\vec{r}}_{=r} + 3r^{n-1}$$

$$= (n-2)r^{n-1} \qquad r \neq -2$$

ex： $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

$$\nabla\cdot(\vec{a}\times\vec{b}) = \partial_i(\vec{a}\times\vec{b})_i$$

$$= \partial_i(\varepsilon_{ijk}a_jb_k)$$

$$= \varepsilon_{ijk}[(\partial_i a_j)b_k + a_j(\partial_i b_k)]$$

$$= \varepsilon_{ijk}b_k\partial_i a_j - \varepsilon_{ijk}a_j\partial_i b_k$$

$$=\vec{b}\cdot(\nabla\times\vec{a})-\vec{a}\cdot(\nabla\times\vec{b})$$

應用數學

Curl（旋度）$\nabla\times\vec{v}$

旋度物理意義：

Circulation（環量）$\equiv\oint\vec{v}\cdot d\vec{l}$　　（$\vec{v}=v_x\hat{i}+v_y\hat{j}+v_z\hat{k}$，$d\vec{l}=dx\hat{i}+dy\hat{j}$）

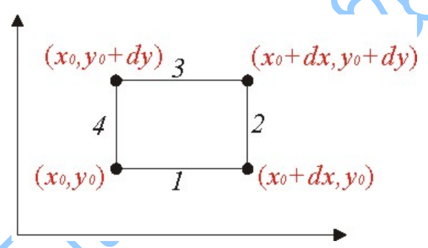
$$=\int_1 v_x(x,y)dx+\int_2 v_y(x,y)dy+\int_3 v_x(x,y)(-dx)+\int_4 v_y(x,y)(-dy)$$

$$\overset{dx,dy\to 0}{=}\ v_x(x,y_0)dx+[v_y(x_0,y)+\frac{\partial}{\partial x}v_y\Big|_{x=x_0}dx+...]$$

$$+[v_x(x,y_0)+\frac{\partial}{\partial y}v_x\Big|_{y=y_0}dy+...]+v_y(x_0,y)(-dy)$$

$$=(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})dxdy$$

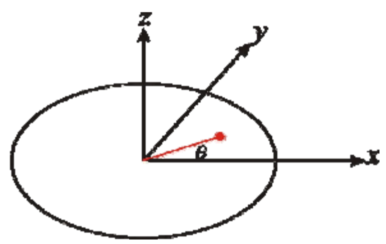
$$=\underbrace{(\nabla\times v)_z}_{\text{旋量vorticity}}dxdy$$




ex：一粒子的平面運動 $\vec{r}=(x,y,0)=(r\cos\theta,r\sin\theta,0)$

$r=$ 定值，$\dot{\theta}=\omega$ 等速率

$$\vec{v}=\frac{d\vec{r}}{dt}=(-r\sin\theta\ddot{\theta},r\cos\theta\ddot{\theta},0)=(-y\omega,x\omega,0)$$




$\nabla\cdot\vec{r}=2$

$\nabla\cdot\vec{v}=0$

$$\nabla\times\vec{r}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix}=0$$

$$\nabla\times\vec{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y\omega & x\omega & 0 \end{vmatrix}=2\omega\hat{k}$$

ex：$\nabla\times(f\vec{v})=f(\nabla\times\vec{v})+\nabla f\times\vec{v}$

$$[\nabla\times(f\vec{v})]_i=\varepsilon_{ijk}\partial_j(f\vec{v})_k=\varepsilon_{ijk}\partial_j(fv_k)$$
$$=\varepsilon_{ijk}[f\partial_j v_k+(\partial_j f)v_k]$$
$$=f(\nabla\times\vec{v})_i+[(\nabla f)\times\vec{v}]_i$$
$$\nabla\times(f\vec{v})=f(\nabla\times\vec{v})+\nabla f\times\vec{v}$$

<另解>：$[\nabla\times(f\vec{v})]_x=\frac{\partial}{\partial y}(fv_z)-\frac{\partial}{\partial z}(fv_y)$

$$=\frac{\partial f}{\partial y}v_z+f\frac{\partial v_z}{\partial y}-\frac{\partial f}{\partial z}v_y-f\frac{\partial v_y}{\partial z}$$
$$=(\nabla f\times\vec{v})_x+f(\nabla\times\vec{v})_x$$

ex：$\nabla\times(f(\vec{r})\vec{r})=0$

$$\nabla\times\vec{r}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix}=0$$

$$\nabla\times(f\vec{r})=f\underbrace{(\nabla\times\vec{r})}_{=0}+\overbrace{\nabla f\times\vec{r}}^{=0}=0$$

ex：$\vec{a}\times(\nabla\times\vec{b})=\nabla_{\vec{b}}(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\nabla)\vec{b}$ （$\vec{b}$ 對b微分）

$$[\vec{a}\times(\nabla\times\vec{b})]_i=\varepsilon_{ijk}a_j(\nabla\times\vec{b})_k$$
$$=\varepsilon_{ijk}\varepsilon_{klm}a_j\partial_l b_m$$
$$=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_j\partial_l b_m$$
$$=a_j\partial_i b_j-a_j\partial_j b_i$$
$$\Rightarrow\ \vec{a}\times(\nabla\times\vec{a})=\frac{1}{2}\nabla(\vec{a}\cdot\vec{a})-(\vec{a}\cdot\nabla)\vec{a}$$

ex：$\nabla(\vec{a}\cdot\vec{b})=(\vec{b}\cdot\nabla)\vec{a}+(\vec{a}\cdot\nabla)\vec{b}+\vec{b}\times(\nabla\times\vec{a})+\vec{a}\times(\nabla\times\vec{b})$

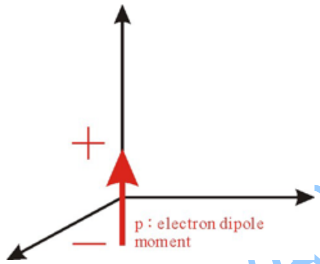
$$\vec{a}\times(\nabla\times\vec{b})=\nabla_{\vec{b}}(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\nabla)\vec{b}\ \ldots\ldots.(1)$$
$$\vec{b}\times(\nabla\times\vec{a})=\nabla_{\vec{a}}(\vec{a}\cdot\vec{b})-(\vec{b}\cdot\nabla)\vec{a}\ \ldots\ldots.(2)$$
$$(1)+(2)\Rightarrow\ \vec{a}\times(\nabla\times\vec{b})+\vec{b}\times(\nabla\times\vec{a})=\nabla(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\nabla)\vec{b}-(\vec{b}\cdot\nabla)\vec{a}$$

ex：$\nabla\times(\vec{a}\times\vec{b})=(\vec{b}\cdot\nabla)\vec{a}-(\vec{a}\cdot\nabla)\vec{b}-\vec{b}(\nabla\cdot\vec{a})+\vec{a}(\nabla\cdot\vec{b})$

$$
\begin{aligned}
[\nabla\times(\vec a\times\vec b)]_i &= \varepsilon_{ijk}\partial_j(\vec a\times\vec b)_k = \varepsilon_{ijk}\partial_j(\varepsilon_{klm}a_l b_m)\\
&= \varepsilon_{ijk}\varepsilon_{klm}\partial_j(a_l b_m)\\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})[(\partial_j a_l)b_m + a_l(\partial_j b_m)]\\
&= (\nabla_j a_i)b_j - (\nabla_j a_j)b_i + a_i(\nabla_j b_j) - a_j(\nabla_j b_i)\\
&= (\vec b\cdot\nabla)a_i - (\nabla\cdot\vec a)b_i + a_i(\nabla\cdot\vec b) - (\vec a\cdot\nabla)b_i\\
&= (\vec b\cdot\nabla)\vec a - (\vec a\cdot\nabla)\vec b - \vec b(\nabla\cdot\vec a) + \vec a(\nabla\cdot\vec b)
\end{aligned}
$$

---

ex：electric dipole $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec P\cdot\vec r}{r^3}$




$$
\begin{aligned}
\vec E &= -\nabla V\\
&= -\nabla\left(\frac{1}{4\pi\varepsilon_0}\frac{\vec P\cdot\vec r}{r^3}\right)\\
&= -\frac{1}{4\pi\varepsilon_0}\nabla\left(\vec P\cdot\frac{\vec r}{r^3}\right)
\end{aligned}
$$

（利用 $\nabla(\vec a\cdot\vec b) = \underbrace{(\vec b\cdot\nabla)\vec a}_{=0} + (\vec a\cdot\nabla)\vec b + \underbrace{\vec b\times(\nabla\times\vec a)}_{=0} + \vec a\times(\nabla\times\vec b)$ ）

$$
\begin{aligned}
&= -\frac{1}{4\pi\varepsilon_0}\left[(\vec P\cdot\nabla)\left(\frac{\vec r}{r^3}\right) + \vec P\times\left(\nabla\times\frac{\vec r}{r^3}\right)\right]\\
&= -\frac{1}{4\pi\varepsilon_0}\left[\frac{\vec P r^2 - 3(\vec P\cdot\vec r)\vec r}{r^5}\right]\\
&= \frac{1}{4\pi\varepsilon_0}\left[\frac{3\hat r(\hat r\cdot\vec p)-\vec P}{r^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[(\vec P\cdot\nabla)\left(\frac{\vec r}{r^3}\right)\right]_i &= (\vec P\cdot\nabla)\left(\frac{r_i}{r^3}\right) = P_j\nabla_j\left(\frac{r_i}{r^3}\right)\\
&= \frac{P_j\delta_{ij}}{r^3} + P_j r_i\left(-\frac{3}{r^4}\frac{r_j}{r}\right)\\
&= \frac{P_i r^2 - 3P_j r_i r_j}{r^5}]
\end{aligned}
$$

---

ex：magnetic dipole $\vec A(r) = \dfrac{\mu_0}{4\pi}\dfrac{\vec m\times\vec r}{r^3}$

$$
\vec B = \nabla\times\vec A = \frac{\mu_0}{4\pi}\nabla\times\left(\vec m\times\frac{\vec r}{r^3}\right) = \frac{\mu_0}{4\pi}\frac{3\hat r(\hat r\cdot\vec m)-\vec m}{r^3}
$$

By $\nabla\times(\vec a\times\vec b) = \underbrace{(\vec b\cdot\nabla)\vec a}_{=0} - (\vec a\cdot\nabla)\vec b - \underbrace{\vec b(\nabla\cdot\vec a)}_{=0} + \vec a(\nabla\cdot\vec b)$

and $\vec{a}=\vec{m}$ , $\vec{b}=\frac{\vec{r}}{r^3}$

ex：two dimensional flow $\vec{V}=U(x,y)\hat{i}-V(x,y)\hat{j}$

imcompressible $\Rightarrow\ \nabla\cdot\vec{V}=0$

irrotational $\Rightarrow\ \nabla\times\vec{V}=0$

$\because\ \nabla\cdot\vec{V}=0 \qquad \frac{\partial U}{\partial x}-\frac{\partial V}{\partial y}=0 \ \Rightarrow\ \frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}$

$\nabla\times\vec{V}=0 \qquad \frac{\partial(-V)}{\partial x}-\frac{\partial U}{\partial y}=0 \ \Rightarrow\ \frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}$ （Couchy-Riemann condition）

$$\frac{\partial}{\partial x}\frac{\partial U}{\partial x}=\frac{\partial}{\partial x}\frac{\partial V}{\partial y}=-\frac{\partial^2 U}{\partial y^2}$$

$$\begin{cases}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})U=0\\(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})V=0\end{cases}$$

ex：$\nabla\cdot\nabla\phi=\nabla^2\phi=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})\phi$

ex：$\nabla\cdot(\nabla\times\vec{V})=0$

$\nabla_i(\nabla\times\vec{V})_i=\nabla_i(\varepsilon_{ijk}\nabla_j V_k)$

$=\varepsilon_{ijk}\nabla_i\nabla_j V_k$ （$=\varepsilon_{jik}\nabla_j\nabla_i V_k$ 　足碼部分 i , j ⇒ j , i，要換要全換）

$=-\varepsilon_{ijk}\nabla_i\nabla_j V_k=0$

ex：$\nabla\times(\nabla\phi)=0$

$[\nabla\times(\nabla\phi)]_i=\varepsilon_{ijk}\nabla_j\nabla_k\phi=0$

ex：$\nabla\times(\nabla\times\vec{V})=\nabla(\nabla\cdot V)-\nabla^2\vec{V}$

$[\nabla\times(\nabla\times\vec{V})]_i=\varepsilon_{ijk}\nabla_j(\nabla\times\vec{V})_k$

$=\varepsilon_{ijk}\nabla_j(\varepsilon_{klm}\nabla_l V_m)$

$=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\nabla_j\nabla_l V_m$

$=\nabla_j\nabla_i V_j-\nabla_j\nabla_j V_i$

$=\nabla(\nabla\cdot V)-\nabla^2\vec{V}$

ex：Maxwel equation

$$\begin{cases} \nabla\cdot\vec{E}\underset{\rho=0}{=}0 & \nabla\times\vec{B}=\varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t} \\ \nabla\cdot\vec{B}=0 & \nabla\times\vec{E}=-\underbrace{\dfrac{\partial\vec{B}}{\partial t}}_{\text{磁場隨時間改變}} \end{cases}$$

（非靜電場，靜電場 $\nabla\times\vec{E}=0$ ）

① $\nabla\times(\nabla\times\vec{E})=-\dfrac{\partial}{\partial t}\nabla\times\vec{B}$

$\Rightarrow\ \nabla(\overset{=0}{\nabla\cdot\vec{E}})-\nabla^2\vec{E}=-\dfrac{\partial}{\partial t}(\varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t})$　　（ $\varepsilon_0\mu_0=\dfrac{1}{c^2}\Rightarrow c(\text{光速})=\dfrac{1}{\sqrt{\varepsilon_0\mu_0}}$ ）

$\Rightarrow\ \nabla^2\vec{E}-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\vec{E}=0$

② $\nabla\times(\nabla\times\vec{B})=\varepsilon_0\mu_0\nabla\times\dfrac{\partial\vec{E}}{\partial t}$

$\Rightarrow\ \nabla(\overset{=0}{\nabla\cdot\vec{B}})-\nabla^2\vec{B}=\dfrac{1}{c^2}\dfrac{\partial}{\partial t}(-\dfrac{\partial\vec{B}}{\partial t})$

$\Rightarrow\ \nabla^2\vec{B}-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\vec{B}=0$

---

line　integral

$\int_c\varphi d\vec{r}$ ， $\int_c\vec{V}\cdot d\vec{r}$ ， $\int_c\vec{V}\times d\vec{r}$

---

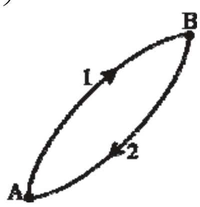
ex：Conservation Force $\int_0^b\vec{F}\cdot d\vec{r}$

① independent of path

② $\oint\vec{F}\cdot d\vec{r}=0$

③ $\vec{F}=-\nabla\phi$

④ $\nabla\times\vec{F}=0$ （stoke's theorem）




$\int_1\vec{F}\cdot d\vec{r}=\int_2\vec{F}\cdot d\vec{r}$

$\Rightarrow\ \int_A^B\vec{F}\cdot d\vec{r}=-\int_B^A\vec{F}\cdot d\vec{r}$

$\Rightarrow\ \oint\vec{F}\cdot d\vec{r}=0$

---

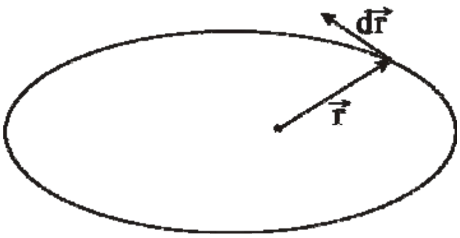
ex： $\vec{F}=(xe^y,ye^z,ze^x)$

$\because\nabla\times\vec{F}\neq0$

$\therefore\vec{F}$ 不守恆

ex：$\oint \vec{r}\times d\vec{r} = ?$ $\quad C=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$，$\vec{r}=(a\cos\theta, b\sin\theta, 0)$

$d\vec{r}=(-a\sin\theta d\theta, b\cos\theta d\theta, 0)$




$$\vec{r}\times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta & b\sin\theta & 0 \\ -a\sin\theta d\theta & b\cos\theta d\theta & 0 \end{vmatrix} = abd\theta\hat{k}$$

$$\oint \vec{r}\times d\vec{r} = \hat{k}ab\int_0^{2\pi} d\theta$$

$$= 2\pi ab\hat{k} = 2\times(\text{橢圓面積})$$

---

surface integral

$\int \varphi d\vec{\sigma}$，$\int \vec{V}\cdot d\vec{\sigma}$，$\int \vec{V}\times d\vec{\sigma}$

ex：$\Phi = \int \vec{B}\cdot d\vec{\sigma}$

$\Phi$：磁通量 magnetic flux

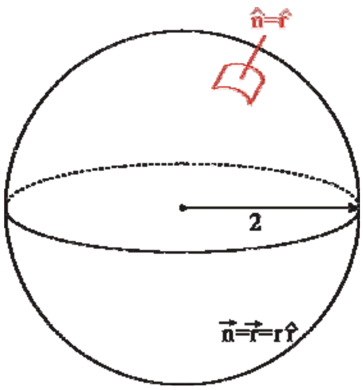
---

ex：$\vec{F}=(x,y,z)$，在 $x^2+y^2+z^2=4$ 上，$\iint \vec{F}\cdot d\vec{\sigma} = ?$

$\vec{F}\cdot d\vec{\sigma} = \vec{F}\cdot\hat{n}d\sigma$

$= (r\hat{r})\cdot\hat{r}d\sigma = rdr$

$\iint \vec{F}\cdot d\vec{\sigma} = \iint rd\sigma = \iint (2)d\sigma = 2\times(4\pi(2)^2) = 32\pi$




---

Volume integral

$\int \vec{V}\cdot d\tau = \hat{i}\int V_x d\tau + \hat{j}\int V_y d\tau + \hat{k}\int V_z d\tau$

$\int \varphi d\tau$

Gauss's Theorem

$\int_V \nabla\cdot\vec{V}d\tau = \int_s \vec{V}\cdot d\sigma$

$\Rightarrow \sum_{six\ surface} \vec{V}\cdot d\vec{\sigma} = \nabla\cdot\vec{V}d\tau$

$\Rightarrow \sum_{exterior} \vec{V}\cdot d\vec{\sigma} = \sum_{volume} \nabla\cdot\vec{V}d\tau$

$\Rightarrow \int_s \vec{V}\cdot d\vec{\sigma} = \int_v \nabla\cdot\vec{V}d\tau$

Alternate forms

（1） $\int_s d\vec{\sigma}\times\vec{P} = \int_v \nabla\times\vec{P}d\tau$

$\vec{V} = \vec{a}\times\vec{P}$ $\quad$ $\vec{a}$ ：任意常數向量

pf： $\oint \vec{a}\times\vec{P}\cdot d\vec{\sigma} = \int \nabla\cdot(\vec{a}\times\vec{p})d\tau$ $\quad$ by $\nabla\cdot(\vec{a}\times\vec{P}) = \vec{P}\cdot(\overset{=0}{\nabla\times\vec{a}}) - \vec{a}\cdot(\nabla\times\vec{P})$

$\Rightarrow \oint \vec{P}\times d\vec{\sigma}\cdot\vec{a} = -\int \vec{a}\cdot(\nabla\times\vec{p})d\tau$

$\Rightarrow \vec{a}\int[\overset{=0}{\vec{P}\times d\vec{\sigma}+\nabla\times\vec{p}d\tau}] = 0$

$\Rightarrow \int_s d\vec{\sigma}\times\vec{P} = \int_v \nabla\times\vec{P}d\tau$

（2） $\int_s \varphi d\vec{\sigma} = \int_v \nabla\varphi d\tau$

$\vec{V} = \varphi\vec{a}$ $\quad$ $\vec{a}$ ：任意常數向量

pf： $\oint_s \varphi\vec{a}\cdot d\vec{\sigma} = \int \nabla\cdot(\varphi\vec{a})d\tau$ $\quad$ by $\nabla\cdot(\varphi\vec{a}) = \nabla\varphi\cdot\vec{a} + \varphi\overset{=0}{\nabla\cdot\vec{a}}$

$\Rightarrow \oint_s \varphi\vec{a}\cdot d\vec{\sigma} = \int(\nabla\varphi)\cdot\vec{a}d\tau$

$\Rightarrow \vec{a}[\overset{=0}{\oint \varphi d\vec{\sigma} - \int \nabla\varphi d\tau}] = 0$

$\Rightarrow \int_s \varphi d\vec{\sigma} = \int_v \nabla\varphi d\tau$

ex：Green's theorem

$\int(U\nabla^2V - V\nabla^2U)d\tau = \int(U\nabla V - V\nabla U)d\vec{\sigma}$

利用 $\nabla\cdot(U\nabla V) = U\nabla^2V + \nabla U\cdot\nabla V$

$\nabla\cdot(V\nabla U) = V\nabla^2U + \nabla V\cdot\nabla U$

兩式相減 $\nabla\cdot(U\nabla V - V\nabla U) = U\nabla^2V - V\nabla^2U$

Stoke's Theorem

$$\oint \vec{V}\cdot d\vec{\lambda} = \int_s \nabla\times\vec{V}\cdot d\vec{\sigma}$$

$$\Rightarrow \sum \vec{V}\cdot d\vec{\lambda} = \nabla\times\vec{V}\cdot d\vec{\sigma}$$ ..........只能用在矩形很小時

所有矩刑相加 $\Rightarrow \sum_{\text{最外圍邊界}} \vec{V}\cdot d\vec{\lambda} = \sum_{\text{所有矩刑}} \nabla\times\vec{V}\cdot d\vec{\sigma}$

$$\Rightarrow \oint \vec{V}\cdot d\vec{\lambda} = \int \nabla\times\vec{V}\cdot d\vec{\sigma}$$

---

ex：magnetic flux

$$\int \vec{B}\cdot d\vec{\sigma} = \int \nabla\times\vec{A}\cdot d\vec{\sigma}$$

$= \oint_c \vec{A}\cdot d\vec{\lambda}$ ..................（if $\vec{A}$ 是常數向量）

$= \Phi$

$= A\cdot(2\pi R)$ ..................（此爲特例！）

---

Gauss's Law

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

$$\oint \vec{E}\cdot d\vec{\sigma} = \oint \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\cdot d\vec{\sigma}$$

$$= \frac{q}{4\pi\varepsilon_0}\int \nabla\cdot(\frac{\hat{r}}{r^2})d\tau$$

$$= -\frac{q}{4\pi\varepsilon_0}\int \nabla^2\frac{1}{r}d\tau$$

$$= \frac{4\pi q}{4\pi\varepsilon_0}\int \delta(\vec{r})d\tau$$

$= \frac{q}{\varepsilon_0}$ ..............（$\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{r})$）

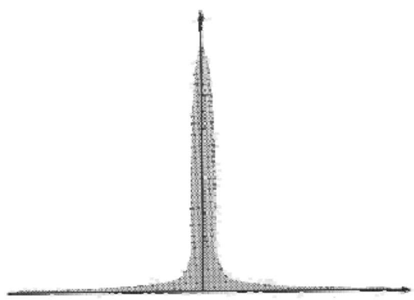
---

Dirac delta function

$$\delta(x) = \begin{cases} \infty, x = 0 \\ 0, x \neq 0 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1$$

pf： $\lim_{\varepsilon\to 0^+} \delta_\varepsilon(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$



$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon}{x^2+\varepsilon^2}dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\frac{x}{\varepsilon}}{1+(\frac{x}{\varepsilon})^2}$$

$$=\frac{1}{\pi}\tan^{-1}\frac{x}{\varepsilon}\Big|_{-\infty}^{\infty}$$

$$=1$$

ex： $x\cdot\delta(x)=0$

ex：點電荷電荷密度 $e(\vec{r})=?$

$$e(\vec{r})=q\delta(\vec{r}-\vec{r}_0)$$

$$\int e(\vec{r})d\tau=\int q\delta(\vec{r}-\vec{r}_0)d\tau$$