Fourier Analysis

$$f(x+2\pi)=f(x) \qquad \text{（period } 2\pi \text{ ）}$$

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}a_0dx+\sum_{n=1}^{\infty}(a_n\underbrace{\int_{-\pi}^{\pi}\cos nxdx}_{=0}+b_n\underbrace{\int_{-\pi}^{\pi}\sin nxdx}_{=0})$$

$$=a_0(2\pi)$$

$$\therefore\ a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\int_{-\pi}^{\pi}f(x)\cos mxdx=\int_{-\pi}^{\pi}a_0\cos mxdx+\sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi}\cos mx\cos nxdx+b_n\int_{-\pi}^{\pi}\cos mx\sin nxdx)$$

$$=\pi a_m$$

$$m\neq n \qquad \int_{-\pi}^{\pi}\cos mx\cos nxdx=\int_{-\pi}^{\pi}\frac{1}{4}[e^{i(m+n)x}+e^{i(-m+n)x}+e^{i(m-n)x}+e^{-i(m+n)x}]dx$$

$$m=n \qquad \int_{-\pi}^{\pi}\cos mx\cos nxdx=\int_{-\pi}^{\pi}\cos^2 mxdx$$

$$=\int_{-\pi}^{\pi}\frac{1+\cos 2mx}{2}dx$$

$$=\pi$$

$$\therefore\ \int_{-\pi}^{\pi}\cos mx\cos nxdx=\pi\delta_{mn}=\int_{-\pi}^{\pi}\sin mx\sin nxdx$$

$$\therefore\ \boxed{a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx}$$

$$\boxed{b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx}$$

$$\boxed{a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx}$$

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eq：square wave $f(x)=\begin{cases}-k & if\ -\pi<x<0\\ k & if\ \ 0<x<\pi\end{cases}$

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}[\int_0^{\pi}kdx+\int_{-\pi}^{0}(-k)dx]$$

$$= \frac{1}{2\pi}[k\pi + (-k)\pi] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}[k\int_0^{\pi}\cos nxdx - k\int_{-\pi}^{0}\cos nxdx]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}[k\int_0^{\pi}\sin nxdx - k\int_{-\pi}^{0}\sin nxdx]$$

$$= \frac{k}{\pi}[\frac{-\cos n\pi + 1}{n} - \frac{-1+\cos n\pi}{n}]$$ （ $\cos n\pi = (-1)^n$ ）

$$= \frac{2k}{n\pi}[1-\cos n\pi]$$

$$= \begin{cases} \frac{4k}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$\therefore \quad f(x) = \frac{4k}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin nx}{n}$$

$$= \frac{4k}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{2m+1}$$

$$f(\frac{\pi}{2}) = k = \frac{4k}{\pi}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots)$$ （ $\because f(x) = \begin{cases} -k & if\ -\pi < x < 0 \\ k & if\ \ 0 < x < \pi \end{cases}$ ）

$$\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \frac{\pi}{4}$$

Any period $p = 2L$

$$f(x+p) = f(x)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L})$$

$$\therefore \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

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eq：a periodic square wave $f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

$$p = 2L = 4 \ \Rightarrow \ L = 2$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx$$

$$=\frac{1}{4}\int_{-1}^{1}k\,dx=\frac{k}{2}$$

$$a_n=\frac{1}{2}\int_{-1}^{1}k\cos\frac{n\pi x}{2}dx=\frac{k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-1}^{1}$$

$$=\frac{k}{n\pi}(\sin\frac{n\pi}{2}-\sin\frac{-n\pi}{2})=\frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n=\frac{1}{2}\int_{-1}^{1}k\sin\frac{n\pi x}{2}dx=0$$

$$f(x)=\frac{k}{2}+\frac{2k}{\pi}(\cos\frac{\pi x}{2}-\frac{1}{3}\cos\frac{3\pi x}{2}+\frac{1}{5}\cos\frac{5\pi x}{2}-...)$$

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eq：Half-wave rectifier $f(t)=\begin{cases}0 & -L<t<0\\ E\sin\omega t & 0<t<L\end{cases}$

$$p=2L=\frac{2\pi}{\omega}\quad\Rightarrow\quad L=\frac{\pi}{\omega}$$

$$a_0=\frac{\omega}{2\pi}\int_0^{\frac{\pi}{\omega}}E\sin\omega t\,dt=\frac{E}{\pi}$$

$$a_n=\frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}}E\sin\omega t\cos\omega t\,dt=\frac{\omega E}{2\pi}\int_0^{\frac{\pi}{\omega}}[\sin(1+n)\omega t+\sin(1-n)\omega t]dt$$

$$=-\frac{2E}{(n^2-1)\pi}\qquad n=2,4,\ldots$$

$$b_n=\frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}}E\sin\omega t\sin\omega t\,dt$$

$b_1=\frac{E}{2}$ ， $b_n=0$ ，n=2,3,…

$$f(x)=\frac{E}{\pi}+\frac{E}{2}\sin\omega t-\frac{2E}{\pi}(\frac{1}{1\cdot 3}\cos 2\omega t+\frac{1}{3\cdot 5}\cos 4\omega t+......)$$

Even and Odd Function

（1） $f(-x)=f(x)$ even function

$\Rightarrow\ b_n=0$

$\Rightarrow\ f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}dx$

（2） $f(-x)=-f(x)$ odd function

$\Rightarrow\ a_0=a_n=0$

$\Rightarrow\ f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}dx$

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eq： $f(x)=x^2$ ， $-\pi<x<\pi$

$\because \quad f(-x)=(-x)^2=f(x)$

$\therefore \quad b_n=0$

$$\Rightarrow \quad f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}dx$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=\frac{\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nxdx=\frac{1}{\pi}[x^2\frac{\sin nx}{n}\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}2x\frac{\sin nx}{n}dx]$$

$$=-\frac{2}{n\pi}\int_{-\pi}^{\pi}x\sin nxdx$$

$$=-\frac{2}{n\pi}[x(-\frac{\cos nx}{n})\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(-\frac{\cos nx}{n})dx]$$

$$=-\frac{2}{n\pi}[\pi(-\frac{\cos n\pi}{n})-(-\pi)(\frac{\cos n\pi}{n})]=\frac{4}{n^2}(-1)^n$$

$$f(x)=x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n4\cos nx}{n^2}$$

$$=\frac{\pi^2}{3}-4(\frac{1}{1^2}\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-......)$$

$$x=0 \quad \Rightarrow \quad 0=\frac{\pi^2}{3}-4(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-......)$$

$$\Rightarrow \quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-......=\frac{\pi^2}{12}$$

$$\Rightarrow \quad \frac{\pi^2}{12}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+......-2(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+......)$$

$$=\xi(2)-\frac{1}{2}\xi(2)$$

$$\therefore \quad \xi(2)=\frac{\pi^2}{6}$$

又 $\xi(4)=\frac{\pi^2}{96}$

$$\xi(n)=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+......=\sum_{p=1}^{n}p^n$$

（ $\xi(n)$ zeta function ）

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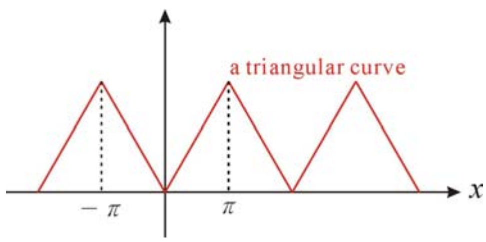
eq： $f(x)=|x|$ ， $-\pi<x<\pi$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}[\int_{0}^{\pi}xdx+\int_{-\pi}^{0}(-x)dx]=\frac{\pi}{2}$$

$$a_n=\frac{2}{\pi}[\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}]=\frac{2}{\pi}[\frac{(-1)^n}{n^2}-\frac{1}{n^2}]$$



$$= \begin{cases} 0 & n \text{ is even} \\ -\dfrac{4}{n^2\pi} & n \text{ is odd} \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...]$$

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eq： $f(x) = |\sin x|$ ， $-\pi < x < \pi$

另外再問 1、$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = ?$ 　　2、$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = ?$

$$a_0 = 2 \cdot \frac{1}{2\pi}\int_0^{\pi} \sin x dx = \frac{1}{\pi}\cos x\Big|_0^{\pi} = \frac{2}{\pi}$$

$$a_n = 2 \cdot \frac{1}{2\pi}\int_0^{\pi} \sin x \cos nx dx$$

$$= \frac{2}{\pi}[\frac{1}{2}\int_0^{\pi} \sin(1+n)x dx + \frac{1}{2}\int_0^{\pi} \sin(1-n)x dx]$$

$$= \frac{2}{\pi}[\frac{-1}{1+n}\cos(1+n)x\Big|_0^{\pi} - \frac{1}{1-n}\cos(1-n)x\Big|_0^{\pi}]$$

$$= \frac{2}{\pi}[(\frac{1}{1+n} + \frac{1}{1-n}) - (\frac{1}{1+n}\cos(1+n)\pi x + \frac{1}{1-n}\cos(1-n)\pi)]$$

$$= \frac{2}{\pi}\frac{1}{1-n^2} \qquad \text{when n=2,4,...}$$

$$\therefore f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2,4,...}^{\infty} \frac{1}{1-n^2}\cos nx$$

$$x = 0 \quad \Rightarrow \quad 0 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{1-4m^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

$$x = \frac{\pi}{2} \quad \Rightarrow \quad 1 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{1-4m^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}$$

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eq： $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ h & 0 < x < \pi \end{cases}$

$f(x+2\pi) = f(x)$

$p = 2L = 2\pi$ ， $L = \pi$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} hdx = \frac{h}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} h\cos nxdx = \frac{h}{\pi}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} h\sin nxdx = -\frac{h}{\pi}\cdot\frac{1}{n}\cos nx\Big|_0^{\pi} = \frac{h}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & n \text{ is even} \\ \frac{2h}{n\pi} & n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{h}{2} + \frac{2h}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin nx$$

應用數學

Half-Range Expansion

（1）全幅：

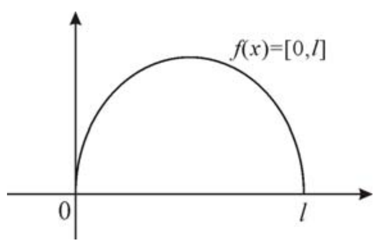
$p = 2L = l$ ， $L = \frac{l}{2}$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{l} + b_n\sin\frac{2n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{2n\pi x}{l}dx$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{2n\pi x}{l}dx$$



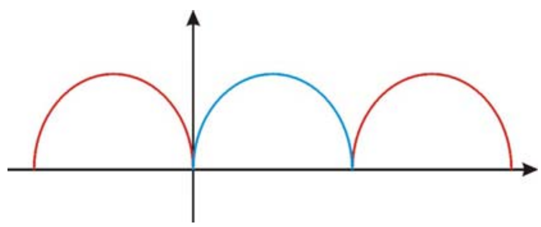
（2）半幅餘弦展開：（偶函數）

$p = 2L = 2l$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}dx$$

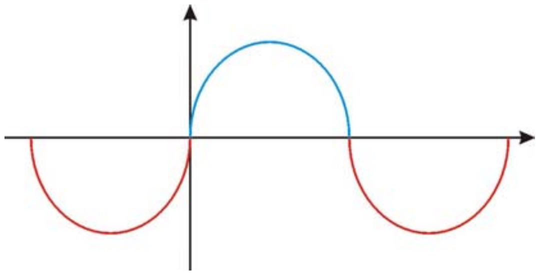


（3）半幅正弦展開：（奇函數）

$$p = 2L = 2l$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$



應用數學

Complex Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= a_0 + \sum_{n=1}^{\infty} [a_n (\frac{e^{inx} + e^{-inx}}{2}) + b_n (\frac{e^{inx} - e^{-inx}}{2})]$$

$$= a_0 + \sum_{n=1}^{\infty} [(\frac{a_n}{2} + \frac{b_n}{2i}) e^{inx} + (\frac{a_n}{2} - \frac{b_n}{2i}) e^{-inx}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

(1)式： $f(x) = a_0 + \sum_{n=1}^{\infty} [c_n e^{inx} + k_n e^{-inx}]$

$$\therefore \quad c_n = \frac{1}{2}(a_n + \frac{b_n}{i}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$k_n = \frac{1}{2}(a_n - \frac{b_n}{i}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx} dx \qquad \text{n=1,2,3,……}$$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\int_{-\pi}^{\pi} f(x) e^{-imx} dx = \sum c_n \int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

$$= \sum_n c_n 2\pi \delta_{nm}$$

$$= 2\pi c_m$$

$$\Rightarrow \boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx}$$

ex： $f(x) = 2\pi x - x^2$ ， $0 \le x \le 2\pi$

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=-\infty}^{\infty} \underbrace{(-\frac{2}{n^2})}_{=c_n} e^{inx} \qquad n \ne 0$$

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} (2\pi x - x^2) dx = \frac{1}{2\pi}[2\pi(\frac{x^2}{2}) - \frac{x^3}{3}]\Big|_0^{2\pi}$$

$$=\frac{1}{2\pi}[4\pi^3-\frac{8\pi^3}{3}]=\frac{2}{3}\pi^2$$

Fourier Integrals（可用來處理非週期性函數）

考慮週期函數 $f(x)$ ， $p=2L$

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\omega_n x+b_n\sin\omega_n x)$$

（ $\omega_n=\frac{n\pi}{L}$ ）

$$=\frac{1}{2L}\int_{-L}^{L}f(x)dx+\frac{1}{L}\sum_{n=1}^{\infty}[\cos\omega_n x(\int_{-L}^{L}f(v)\cos\omega_n vdv)+\sin\omega_n x(\int_{-L}^{L}f(v)\sin\omega_n vdv)]$$

$$\Delta\omega=\omega_{n+1}-\omega_n=\frac{(n+1)\pi}{L}-\frac{n\pi}{L}=\frac{\pi}{L}$$

$L\to\infty$ ， $\Delta\omega\to d\omega$

$$\sum_{n=-\infty}^{\infty}\ \to\int_{-\infty}^{\infty}\ \Rightarrow\sum_{n=1}^{\infty}\ \to\int_{0}^{\infty}$$

$\omega_n\to\omega$

$$f(x)\underset{L\to\infty}{=}\frac{1}{\pi}\int_0^{\infty}[\cos\omega x(\int_{-\infty}^{\infty}f(v)\cos\omega vdv)+\sin\omega x(\int_{-\infty}^{\infty}f(v)\sin\omega vdv)]d\omega$$

$$\Rightarrow\ f(x)=\int_0^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\cos\omega vdv$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\sin\omega vdv$$

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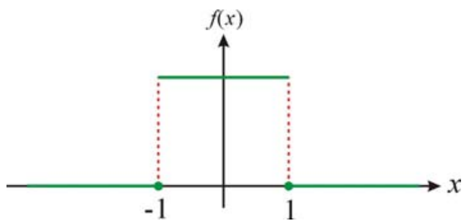
ex：single pulse 脈衝

$$f(x)=\begin{cases}1 & |x|<1\\ 0 & |x|>1\end{cases}$$



$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\cos\omega vdv$$

$$=\frac{1}{\pi}\int_{-1}^{1}\cos\omega vdv=\frac{1}{\pi}\cdot\frac{1}{\omega}\sin\omega v\Big|_{-1}^{1}=\frac{2\sin\omega}{\pi\omega}$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\sin\omega vdv$$

$$=\frac{1}{\pi}\int_{-1}^{1}\sin\omega vdv=0$$

$$f(x)=\int_0^{\infty}(\frac{2\sin\omega}{\pi\omega})\cos\omega xd\omega=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}d\omega$$

$x=0 \qquad \dfrac{2}{\pi}\int_0^\infty \dfrac{\sin\omega}{\omega}d\omega = 1 \Rightarrow \boxed{\int_0^\infty \dfrac{\sin\omega}{\omega}d\omega = \dfrac{\pi}{2}}$

$0 \le x < 1 \qquad 1 = \dfrac{2}{\pi}\int_0^\infty \dfrac{\sin\omega\cos\omega x}{\omega}d\omega \Rightarrow \int_0^\infty \dfrac{\sin\omega\cos\omega x}{\omega}d\omega = \dfrac{\pi}{2}$

$x > 1 \qquad 0 = \dfrac{2}{\pi}\int_0^\infty \dfrac{\sin\omega\cos\omega x}{\omega}d\omega \Rightarrow \int_0^\infty \dfrac{\sin\omega\cos\omega x}{\omega}d\omega = 0$

$x = 1 \qquad \int_0^\infty \dfrac{\sin\omega\cos\omega}{\omega}d\omega = \dfrac{1}{2}\int_0^\infty \dfrac{\sin 2\omega}{2\omega}d(2\omega) = \dfrac{\pi}{4}$

$$\therefore \int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega = \begin{cases} \dfrac{\pi}{2} & 0 \le x < 1 \\ \dfrac{\pi}{4} & x = 1 \\ 0 & x > 1 \end{cases}$$

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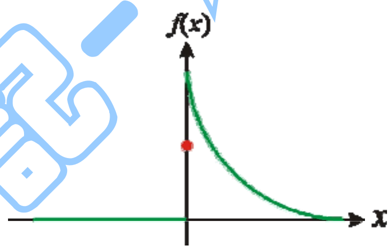
ex： $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$



$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos\omega v dv = \frac{1}{\pi}\int_{-\infty}^\infty e^{-v}\cos\omega v dv$$

$$\begin{aligned}\int_{-\infty}^\infty e^{-v}\cos\omega v dv &= e^{-v}\frac{\sin\omega v}{\omega}\Big|_0^\infty - \int_0^\infty (-e^{-v})(\frac{\sin\omega v}{\omega})dv \\ &= \frac{1}{\omega}\int_0^\infty -e^{-v}\sin\omega v dv \\ &= \frac{1}{\omega}[e^{-v}(-\frac{\cos\omega v}{\omega})\Big|_0^\infty - \int_0^\infty (-e^{-v})(\frac{\cos\omega v}{\omega})dv] \\ &= \frac{1}{\omega}[\frac{1}{\omega} - \frac{1}{\omega}\int_0^\infty -e^{-v}\cos\omega v dv]\end{aligned}$$

$$= \frac{1}{1+\omega^2}$$

$$A(\omega) = \frac{1}{\pi} \cdot \frac{1}{1+\omega^2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv = \frac{1}{\pi} \cdot \frac{\omega}{1+\omega^2}$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}]d\omega$$

$$x = 0 \quad f(0) = \frac{1}{\pi}\int_0^{\infty}[\frac{1}{1+\omega^2}]d\omega$$

$$= \frac{1}{\pi}\tan^{-1}\omega\Big|_0^{\infty} = \frac{1}{2}$$

$$f(0) = \frac{1}{2}[f(0_+) + f(0_-)]$$

Dirichlet 定理：

$$f(x_0) = \frac{1}{2}[f(x_{0+}) + f(x_{0-})]$$

應用數學

Fourier Transform

$$f(x) = \int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)dv]d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\underbrace{\sin(\omega x - \omega v)}_{\omega\text{的奇函數}}dvd\omega = 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)dvd\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}dvd\omega \text{ .... }$$ （complex Fourier integral）

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \equiv \mathcal{F}(f)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\omega)e^{i\omega x}d\omega \text{ .... }$$ （inverse Fourier transform）

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{i(\omega x-\omega v)}dvd\omega$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}dx$$

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ex： $f(x)=1$

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}dx$$

$$= \sqrt{2\pi}\,\delta(\omega)$$

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ex： $f(x)=e^{-\frac{x^2}{2}}$

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2i\omega x)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x+i\omega)^2+\omega^2]}dx = e^{-\frac{1}{2}\omega^2}$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_{0}^{\infty} e^{-x^2}dx$$

$$= 2\int_{0}^{\infty} e^{-y}\cdot\frac{1}{2}y^{-\frac{1}{2}}dy$$

$$= \Gamma(\frac{1}{2})$$

$$= \sqrt{\pi}$$

$y \equiv x^2 \Rightarrow dy = 2xdx$

$\Rightarrow dx = \frac{1}{2}y^{\frac{1}{2}}dy$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx = \sqrt{2}\int_{-\infty}^{\infty} e^{-(\frac{x}{\sqrt{2}})^2}d(\frac{x}{\sqrt{2}})$$

$$= \sqrt{2\pi}$$

1、$\mathcal{F}\{af(x)+bg(x)\} = a\mathcal{F}\{f(x)\}+b\mathcal{F}\{g(x)\}$　　（linearity）

2、$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}$

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx = i\omega\mathcal{F}\{f(x)\}$$

$$\mathscr{F}\{f''(x)\}=(i\omega)^2\mathscr{F}\{f(x)\}$$

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ex： $f(x)=xe^{-x^2}$ $\qquad \mathscr{F}\{e^{-ax^2}\}=\frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}$

$$\mathscr{F}\{f(x)\}=\mathscr{F}\{xe^{-x^2}\}$$
$$=\mathscr{F}\{-\frac{1}{2}e^{-x^2}\}$$
$$=-\frac{1}{2}(i\omega)\mathscr{F}\{e^{-x^2}\}$$
$$=-\frac{i\omega}{2\sqrt{2}}e^{-\frac{\omega^2}{4}}$$

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ex： $\mathscr{F}\{\underbrace{\int_{-\infty}^{t}f(\tau)d\tau}_{=g(x)}\}=\frac{1}{i\omega}\mathscr{F}\{f(x)\}$

$$f(x)=g'(x)$$
$$\mathscr{F}\{f(x)\}=\mathscr{F}\{g'(x)\}=(i\omega)\mathscr{F}\{g(x)\}$$
$$\Rightarrow\ \mathscr{F}\{g(x)\}=\frac{1}{i\omega}\mathscr{F}\{f(x)\}$$

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ex： $y'-4y=\theta(x)e^{-4x}$ $\qquad \theta(x)=\begin{cases}1 & x\ge 0\\ 0 & x<0\end{cases}$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\theta(x)e^{-4x}e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\theta(x)e^{-(4+i\omega)x}dx$$
$$=\frac{1}{\sqrt{2\pi}}\frac{-e^{-(4+i\omega)x}}{4+i\omega}\Bigg|_0^{\infty}$$

$$i\omega Y-4Y=\frac{1}{4+i\omega}\frac{1}{\sqrt{2\pi}}$$
$$Y(\omega)=-\frac{1}{\sqrt{2\pi}}\frac{1}{16+\omega^2}$$
$$y(x)=\mathscr{F}^{-1}\{-\frac{1}{\sqrt{2\pi}}\frac{1}{16+\omega^2}\}=\begin{cases}-\frac{1}{8}e^{4x} & x<0\\ -\frac{1}{8}e^{-4x} & x\ge 0\end{cases}$$

$$\left(\ \frac{1}{8i}\left(\frac{1}{\omega+4i}-\frac{1}{\omega-4i}\right)\frac{1}{\sqrt{2\pi}}=-\frac{1}{8}\left(\frac{1}{4-i\omega}+\frac{1}{4+i\omega}\right)\frac{1}{\sqrt{2\pi}}\ \right)$$

（ $e^{-ax}=\sqrt{\frac{2}{\pi}}(\frac{a}{a^2+\omega^2})$ ， $a>0$ ）

Convolution（卷積）

$$(f*g)(x)=\int_{-\infty}^{\infty}f(t)g(x-t)dt$$
$$=\int_{-\infty}^{\infty}f(x-t)g(x)dt$$

$$\Rightarrow \mathcal{F}\{f*g\}=\sqrt{2\pi}\mathcal{F}\{f\}\mathcal{F}\{g\}$$

pf： $\mathcal{F}\{f*g\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)g(x-t)e^{-i\omega x}dtdx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)g(\underbrace{x-t}_{=q})e^{-i\omega x}dxdt \qquad x-t=q \Rightarrow x=t+q$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)g(q)e^{-i\omega(t+q)}dqdt \qquad \Rightarrow dx=dq$$

$$=\frac{1}{\sqrt{2\pi}}(\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt)(\int_{-\infty}^{\infty}g(q)e^{-i\omega q)}dq)$$

$$=\sqrt{2\pi}\mathcal{F}\{f\}\mathcal{F}\{g\}$$

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ex： $\mathcal{F}^{-1}\{\frac{5}{2-\omega^2+3i\omega}\}=\mathcal{F}^{-1}\{(\frac{5}{2+i\omega})(\frac{1}{1+i\omega})\}$

$$=(\sqrt{2\pi}\cdot 5\theta(x)e^{-2x})*(\sqrt{2\pi}\theta(x)e^{-x})\frac{1}{\sqrt{2\pi}}$$

$$\mathcal{F}^{-1}\{f\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}5\theta(x)e^{-2x}e^{-i\omega x}dx$$
$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}5e^{-(2+i\omega)x}dx$$
$$=\frac{1}{\sqrt{2\pi}}(-\frac{5e^{-(2+i\omega)x}}{2+i\omega})\Big|_0^{\infty}$$
$$=\frac{1}{\sqrt{2\pi}}\frac{5}{2+i\omega}$$

$$=\sqrt{2\pi}\cdot 5\int_{-\infty}^{\infty}\theta(t)e^{-2t}\theta(x-t)e^{-(x-t)}dt$$
$$=5\sqrt{2\pi}\int_{0}^{\infty}\theta(x-t)e^{-(x+t)}dt$$
$$=5\sqrt{2\pi}e^{-x}\int_{0}^{\infty}\theta(x-t)e^{-t}dt$$

$$= 5\sqrt{2\pi}e^{-x}\int_0^x e^{-t}dt$$

$$= 5\sqrt{2\pi}e^{-x}(-e^{-x}+1)\ \ldots\ldots\ x \ge 0$$

$$= \begin{cases} 5\sqrt{2\pi}e^{-x}(1-e^{-x}) & x \ge 0 \\ 0 & x < 0 \end{cases}$$

應用數學