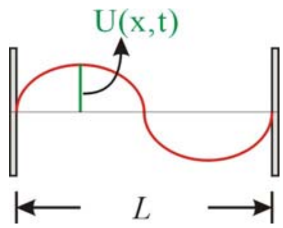
Partial Differential Equation

(1) 一維波方程 $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}=\frac{\partial^2 U}{\partial x^2}$ amplitude



$U(0,t)=U(L,t)=0$……… （Boundary condition）

$U(x,0)=f(x)$（波型），$\left.\frac{\partial U}{\partial t}\right|_{t=0}=g(x)$（起始速度）……（inutual condition）

$U(x,t)=F(x)G(t)$

$$G\frac{d^2F}{dx^2}-\frac{1}{c^2}F\frac{d^2G}{dt^2}=0$$

$$\underbrace{\frac{1}{F}\frac{d^2F}{dx^2}}_{x\text{函數}}=\underbrace{\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}}_{t\text{函數}}=\underbrace{-p^2}_{\text{常數}}$$

（p：視邊界條件而設 $p>0$，當然，p 也可以 <0，但會很麻煩。）

（兩個不同函數要相等，除非兩者皆爲常數）

$\therefore$ $\boxed{\frac{d^2F}{dx^2}+p^2F=0}$ 或 $\boxed{\frac{d^2G}{dt^2}+c^2p^2G=0}$

① $\frac{d^2F}{dx^2}+p^2F=0 \Rightarrow F=A\cos px+B\sin px$

$\Rightarrow \begin{cases}F(0)=0\\F(L)=0\end{cases} \Rightarrow \begin{matrix}A=0\\F=B\sin pL\end{matrix} \Rightarrow pL=n\pi \quad n=1,2,3,\ldots$

$\Rightarrow \boxed{p=\frac{n\pi}{L}}$

$\therefore \boxed{F_n\propto\sin\frac{n\pi x}{L}}$

② $\frac{d^2G}{dt^2}+c^2p^2G=0 \Rightarrow G=C\cos\lambda t+D\sin\lambda t$

$\Rightarrow \lambda^2=c^2p^2$

$\therefore \boxed{\lambda=\lambda_n=\frac{cn\pi}{L}}$

$$U=\sum_n U_n=\sum_n (C_n\cos\lambda_n t+D_n\sin\lambda_n t)\sin\frac{n\pi x}{L}$$

代入邊界條件

$$U(x,0)=f(x)\ \Rightarrow\ f(x)=\sum_n C_n\sin\frac{n\pi x}{L}$$

$$\Rightarrow \int_0^L f(x)\sin\frac{n'\pi x}{L}dx=\int_0^L\sum_n C_n\sin\frac{n'\pi x}{L}\sin\frac{n\pi x}{L}dx$$

$$=\sum_n C_n\frac{L}{2}\delta_{nn'}$$

$$=\frac{L}{2}C_{n'}$$

$$\therefore\ \boxed{C_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=0}=g(x)\ \Rightarrow\ \left.\frac{\partial U}{\partial t}\right|_{t=0}=\left.\sum_n(-C_n\lambda_n\sin\lambda_n t+D_n\lambda_n\cos\lambda_n t)\sin\frac{n\pi x}{L}\right|_{t=0}$$

$$=\sum_n D_n\lambda_n\sin\frac{n\pi x}{L}=g(x)$$ （兩邊同乘上 $\int_0^L\sin\frac{n'\pi x}{L}dx$ ）

$$\Rightarrow \sum_n D_n\lambda_n\frac{L}{2}\delta_{nn'}=\int_0^L g(x)\sin\frac{n'\pi x}{L}dx=D_n'\lambda_n'\frac{L}{2}$$

$$\Rightarrow \boxed{D_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx}$$

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(2) one-dimension heat equation

$$\frac{\partial U}{\partial t}=c^2\frac{\partial^2 U}{\partial x^2}\text{，}U(x,t)\text{，}c^2=\frac{\kappa}{\sigma\rho}$$

$$U(0,t)=0\text{，}U(L,t)=0\text{，}U(x,0)=f(x)$$

$$U(x,t)=F(x)G(t)$$

$$\Rightarrow F\dot{G}=c^2GF''$$

$$\Rightarrow \frac{\dot{G}}{c^2G}=\frac{F''}{F}=-p^2\text{，}\ p>0$$

① $F''+p^2F=0$

$F=A\cos px+B\sin px$

=0,由邊界條件得知

$F(L)=0=B\sin pL$ $\boxed{pL=n\pi}$，$n=1,2,3,\ldots$

$$\boxed{F_n=\sin\frac{n\pi x}{L}}$$

② $\dot{G}+p^2c^2F=0$

$$\Rightarrow \quad G = e^{-p^2c^2t}$$

$$\Rightarrow \quad \boxed{G_n = D_n e^{-\lambda_n^2 t}} \text{，} \boxed{\lambda_n = \frac{n\pi c}{L}}$$

$$\therefore \quad \boxed{U(x,t) = \sum_n D_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}}$$

$$U(x,0) = f(x) = \sum_n D_n \sin\frac{n\pi x}{L}$$

$$\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{L}{2}D_n$$

$$\therefore \quad \boxed{D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}$$

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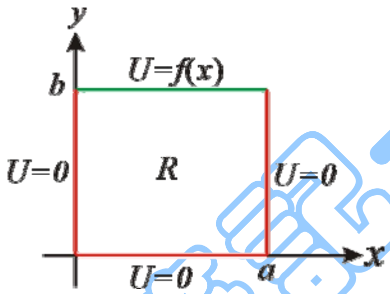
(3) Steady-state two dimension heat flow

$$\frac{\partial U}{\partial t} = c^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$

$\because$ steady（穩定不變的） $\quad\quad \therefore \frac{\partial U}{\partial t} = 0$



$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$$U(x,y) = F(x)G(y)$$

$$G\frac{d^2F}{dx^2} + F\frac{d^2G}{dy^2} = 0 \quad \Rightarrow \quad \frac{1}{F}\frac{d^2F}{dx^2} + \frac{1}{G}\frac{d^2G}{dy^2} = 0$$

$$\Rightarrow \quad \frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -k^2$$

① $\frac{d^2F}{dx^2} + k^2F = 0$

$$F_n = \sin\frac{n\pi x}{a} \quad\quad ka = n\pi \text{，} k = \frac{n\pi}{a}$$

② $\frac{d^2G}{dy^2} - k^2G = 0$

$$G_n = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$$

$$G_n(y=0) = 0 = A_n + B_n$$

$$G_n(y) = \underbrace{2A_n}_{=A_n^*} \sinh\frac{n\pi y}{a}$$

$$U(x,y) = \sum_n A_n^* \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

如圖， $f(x) = (x,b) = \sinh\frac{n\pi b}{a} \sum_n A_n^* \sin\frac{n\pi x}{a}$

$$A_n^* = \frac{2}{a\sinh\frac{n\pi b}{a}} \int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

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(4) two-dimension wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$

$U(t<0)=0$ ， $U(x,y,0) = f(x,y)$ ， $\left.\frac{\partial U}{\partial t}\right|_{t=0} = g(x,y)$

$$U(x,y,t) = F(x,y)G(t)$$

$$F\ddot{G} = c^2 G\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$$

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{F}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) = -\nu^2$$

$\boxed{\ddot{G} + \lambda^2 G = 0}$ …… $\lambda = c\nu$

$$G = B\cos\lambda t + B^*\sin\lambda t$$

$\boxed{\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \nu^2 F = 0}$ （Helmholtz's eq）

$$F(x,y) = H(x)Q(y)$$

$$\frac{1}{H}\frac{d^2 H}{dx^2} = -\left(\frac{1}{Q}\frac{d^2 Q}{dy^2} + \nu^2 Q\right) = -k^2$$

$\boxed{H_m(x) = \sin\frac{m\pi x}{a}}$ ， $k = \frac{m\pi}{a}$

$$\frac{d^2 Q}{dy^2} + \underbrace{(\nu^2 - k^2)}_{=l^2 = (\frac{n\pi}{n})^2} Q = 0 \Rightarrow Q_n(y) = \sin\frac{n\pi y}{b}$$

$$\nu^2 - k^2 = \left(\frac{n\pi}{b}\right)^2 \Rightarrow \nu^2 = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

$$\lambda = cv \quad \Rightarrow \quad \lambda_{mn} = cv_{mn} = c\pi\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

$$\boxed{G_{mn} = B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t}$$

$$U(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

邊界條件： $U(0,y,t) = U(a,y,t) = 0$

$U(x,0,t) = U(x,b,t) = 0$

$U(t<0) = 0$

$U(x,y,0) = f(x,y)$

$\left.\frac{\partial U}{\partial t}\right|_{t=0} = g(x,y)$

可求得上式

$$U(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\int_0^b\int_0^a f(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dxdy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\int_0^b\int_0^a \underbrace{\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}_{(\frac{a}{2}\delta_{m'm})(\frac{b}{2}\delta_{n'n})}f(x,y)dxdy$$

$$= \frac{ab}{4}B_{m'n'}$$

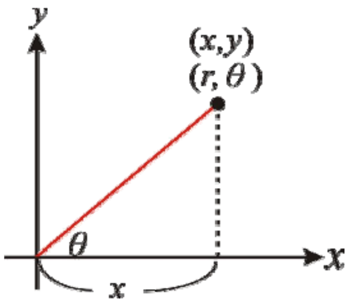
$$\boxed{B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy}$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}^*\lambda_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x,y)$$

$$\boxed{B_{mn}^* = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy}$$

Laplacian in polar coordinates（ r,θ ）

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases}$$



$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

$$\frac{\partial U}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial U}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial U}{\partial \theta} = \cos\theta \frac{\partial U}{\partial r} - \frac{\sin\theta}{r}\frac{\partial U}{\partial \theta}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} = \frac{-y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

$$r_x = \frac{x}{r}$$

$$r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}$$

$$\theta_x = \frac{-y}{r^2}$$

$$\theta_{xx} = -y(-\frac{2}{r^3})r_x = \frac{2xy}{r^4}$$

(5) Circular Membrane（薄膜）

$$\frac{\partial^2 U}{\partial t^2} = c^2 \nabla^2 U$$

邊界條件：$U(R,t) = 0$

$$U(r,0)=f(r)$$

$$\left.\frac{\partial U}{\partial t}\right|_{t=0}=g(r)$$

$$\frac{\partial^2 U}{\partial t^2}=c^2\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right)$$

$U=U(r,t)=W(r)G(t)$ ……radially（徑向）（沿 r 方向）symmetric

$$\frac{\ddot{G}}{c^2 G}=\frac{1}{W}\left(W''+\frac{1}{r}W'\right)=-k^2 \qquad (\ k>0\ )$$

$$\boxed{\ddot{G}+\lambda^2 G=0} \qquad \lambda^2=c^2k^2$$

$$\boxed{W''+\frac{1}{r}W'+k^2W=0}$$

$$\left(\ x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0\ \right)$$

$$r^2\frac{d^2W}{dr^2}+r\frac{dW}{dr}+k^2r^2W=0$$

$kr\equiv x$ ， $\omega=J_0(kr)$

$$U(R,t)=0 \ \Rightarrow\ J_0(KR)=0$$

$$KR=\alpha_{0m}$$

$$\boxed{K_m=\frac{\alpha_{0m}}{R}} \qquad \boxed{\omega_m(r)=J_0\left(\frac{\alpha_{0m}r}{R}\right)}$$

$$G_m(t)=a_m\cos\lambda_m t+b_m\sin\lambda_m t$$

$$U(r,t)=\sum_{m=1}^{\infty}(a_m\cos\lambda_m t+b_m\sin\lambda_m t)J_0\left(\frac{\alpha_m r}{R}\right)$$

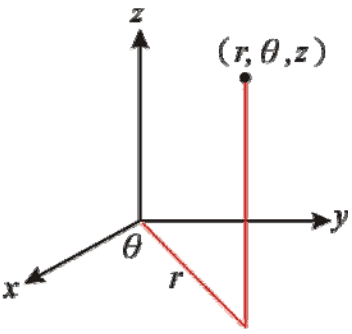
$$U(r,0)=f(r)=\sum_{m=1}^{\infty}a_m+J_0\left(\frac{\alpha_m r}{R}\right)$$

$$\int_0^R xJ_0^2(k_{m0}x)dx=\frac{R^2}{2}J_1^2(k_{m0}R)$$

$J_m(\underbrace{k_{mn}R}_{\alpha_{mn}})=0$ ， $J_1(\alpha_{0m})\neq 0$

$$a_m=\frac{2}{R^2J_1^2(\alpha_m)}\int_0^R rf(r)J_0\left(\frac{\alpha_m r}{R}\right)dr$$

Laplacian in cylindrical coordinates（ r,θ,z ）



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$r > 0$ ， $-\pi < \theta \le \pi$ ， $-\infty < z < \infty$

$r = \sqrt{x^2 + y^2}$

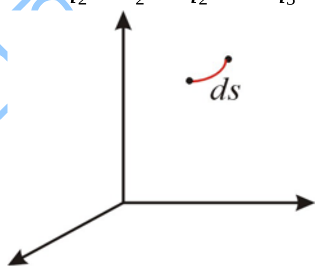
$\theta = \tan^{-1}\dfrac{y}{x}$

$z = z$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}$$

$$\nabla^2 U = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial U}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial U}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial U}{\partial q_3}\right)\right] \ldots\ldots(1)$$



$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2 + dz^2$$

$$= dr^2 + r^2 d\theta^2 + dz^2$$

$$= (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

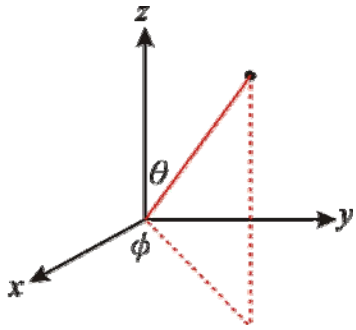
h_1 ， h_2 ， h_3 ：scale factor $h_1 = 1$ ， $h_2 = r$ ， $h_3 = 1$代入(1)式可得

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}$$

Spherical Coordinates（ r, θ, ϕ ）

$r > 0$ ， $0 \le \theta < \pi$ ， $0 \le \phi < 2\pi$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



$$x^2 + y^2 + z^2 = r^2$$

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (dr\sin\theta\cos\phi + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi)^2 + (...)^2 + (...)^2 \\ &= (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2 \\ &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \end{aligned}$$

$\therefore\ h_1 = 1$，$h_2 = r$，$h_3 = r\sin\theta$ 代入

$$\begin{aligned} \nabla^2 U &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial U}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial U}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial U}{\partial q_3}\right)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}\right)\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} \end{aligned}$$

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ex：$U = U(r)$

$$\begin{aligned} \nabla^2 U &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) \\ &= \frac{1}{r}\frac{d^2}{dr^2}(rU) \qquad \left(\frac{1}{r}\frac{d}{dr}\left(U + r\frac{dU}{dr}\right) = \frac{1}{r}\frac{dU}{dr} + \frac{1}{r}\frac{dU}{dr} + \frac{d^2U}{dr^2}\right) \\ &= \frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} \end{aligned}$$

$(\nabla^2 + k^2)U = 0$

$U = e^{i\vec{k}\cdot\vec{r}}$ （plane wave）

$U = \dfrac{e^{ikr}}{r}$ （球面波）

Laplacian

$\nabla^2 U = 0$

$U = U(r,\theta)$ ……U 和 ϕ 無關（azimuthal symmetry）方位角 ϕ

$$U(r,\theta) = R(r)\Theta(\theta)$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial U}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial U}{\partial\theta}) = 0$$
$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) = l(l+1)$$ ……（l：在量子力學中爲角動量）

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$$

令 $R \sim r^\alpha$

$$\alpha(\alpha-1) + 2\alpha - l(l+1) = 0$$
$$\alpha(\alpha+1) - l(l+1) = 0$$
$$\alpha = l, -(l+1)$$
$$\therefore\ R \sim A_l r^l + B_l r^{-(l+1)}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + l(l+1)\Theta = 0$$
$$\cos\theta \equiv \omega$$
$$(-\sin\theta)d\theta = d(\cos\theta)$$
$$(1-\omega^2)\frac{d^2P}{d\omega^2} - 2\omega\frac{dP}{d\omega} + l(l+1)P = 0$$
$$\Theta = P_l(\omega)$$

$$U = \sum_l (A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta)$$

$$\int_{-1}^{1} P_l(\omega)P_{l'}(\omega)d\omega = \frac{2}{2l+1}\delta_{ll'}$$