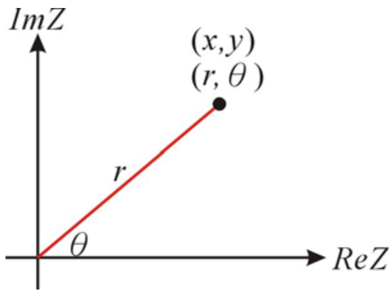
Complex Analysis

$$
\begin{aligned}
z &= x + iy \\
&= re^{i\theta} \\
&= r\cos\theta + ir\sin\theta
\end{aligned}
$$



$r^2 = x^2 + y^2$，$r = \sqrt{x^2 + y^2} = |z|$，$\theta = \tan^{-1}\dfrac{y}{x}$

$-\pi < Argz < \pi$（主值：principle value）

$z^* = x - iy$（complex conjugate of z，共軛複數）

De Moivre's Formula

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$(e^{i\theta})^n = e^{in\theta}$$

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nth root（根）

$$\sqrt[n]{z} = \omega$$

$$Z = \omega^n = r(\cos\theta + i\sin\theta)$$

$$\omega = r^{\frac{1}{n}}[\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}]，k = 0,1,2,\ldots,n-1$$

$$n = 3 \;\Rightarrow\; \omega^3 = Z = 1 \;\Rightarrow\; \omega = 1$$

$$\therefore\; \omega = \cos\frac{\theta + 2\pi k}{3} + i\sin\frac{\theta + 2\pi k}{3}$$

$$\therefore\; \theta = 0 \;\Rightarrow\; \omega = 1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$$

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ex：$e^{\frac{1}{3}}$

$$i = e^{i\frac{\pi}{2}} \;\Rightarrow\; e^{\frac{1}{3}} = e^{i(\frac{\pi}{2} + 2\pi k)\frac{1}{3}}，k = 0,1,2.$$

$$= e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{3\pi}{2}}$$

ex： $\ln(-1)$

$\ln(-1)=\ln(e^{i(\pi+2\pi n)})=i(\pi+2\pi n)$ ， $n\in Z$

ex： i^i

$i^i=e^{\ln i^i}=e^{i\ln i}=e^{i\ln e^{i\frac{\pi}{2}}}=e^{i(i\frac{\pi}{2})}=e^{-\frac{\pi}{2}}$

ex： $3^{\frac{1}{i}}$

$3^{\frac{1}{i}}=3^{-i}=e^{\ln 3^{-i}}=e^{-i\ln 3}=\cos\ln 3-i\sin\ln 3$

應用數學

Complex function

$f(z)=u(x,y)+iv(x,y)$

$z=x+iy$ （complex number）

ex： e^z

$$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$$
$$=e^x\cos y+ie^x\sin y$$
$$=\sin x\cosh y+i\cos x\sinh y$$

$$\cos iy=\frac{e^{i(iy)}+e^{-i(iy)}}{2}=\frac{e^{-y}+e^{y}}{2}=\cosh y$$

$$\sin iy=\frac{e^{i(iy)}-e^{-i(iy)}}{2i}=i\sinh y$$

$$\therefore\ \sin z=\frac{e^{iz}-e^{-iz}}{2i}=\sin(x+iy)=\sin x\cos iy+\cos x\sin iy$$

$$\cos z=\frac{e^{iz}+e^{-iz}}{2}=\cos x\cosh y-i\sin x\sinh y$$

$$\sinh z=\frac{e^{z}-e^{-z}}{2}=\sinh x\cos y+i\cosh x\sin y$$

$$\cosh z=\frac{e^{z}+e^{-z}}{2}=\cosh x\cos y+i\sinh x\sin y$$

ex： $\ln z$ 　多値函數

$\ln z=\ln[re^{i(\theta+2\pi n)}]=\ln r+i(\theta+2\pi n)$

$n=0\ \Rightarrow$ 主値 $\ln z=\ln r+i\theta$

（ $\ln z_1z_2\neq\ln z_1+\ln z_2$ ）

ex： $\sin^{-1} z = \frac{1}{i}\ln(iz \pm \sqrt{1-z^2})$

$$\sin^{-1} z = \omega \;\Rightarrow\; z = \sin\omega = \frac{e^{i\omega}-e^{-i\omega}}{2i}$$

$$\Rightarrow\; e^{i\omega}-e^{-i\omega} = 2iz$$

$$\overset{*e^{i\omega}}{\Rightarrow}\; e^{2i\omega}-2ize^{i\omega}-1=0$$

$$\overset{e^{i\omega}=t}{\Rightarrow}\; t^2-2izt-1=0$$

$$\Rightarrow\; t=\frac{2iz\pm\sqrt{-4z^2+4}}{2} = iz\pm\sqrt{1-z^2}$$

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ex： $\cos^{-1} z = \frac{1}{i}\ln(z \pm \sqrt{z^2-1})$

$$\tan^{-1} z = \frac{1}{2i}\ln\frac{1+iz}{1-iz}$$

$$\sinh^{-1} z = \ln(z+\sqrt{z^2+1})$$

$$\cosh^{-1} z = \ln(z+\sqrt{z^2-1})$$

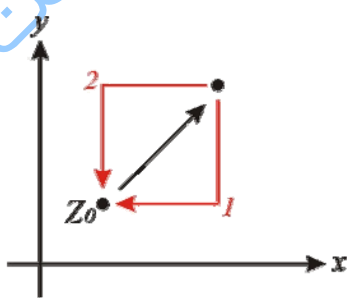
$$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$$

應用數學

Analytic Function

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = \left.\frac{df}{dz}\right|_{z=z_0}$$

$$z = x+iy \;\Rightarrow\; \Delta z = \Delta x + i\Delta y$$



(1) $\Delta y = 0$（沿 x 軸趨近於 z_0 ）

$$\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = \lim_{\Delta x\to 0}\frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

（ $f = u+iv$ ）

(2) $\Delta x = 0$（沿 y 軸趨近於 z_0 ）

$$\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = \lim_{\Delta y\to 0}\frac{\Delta u + i\Delta v}{i\Delta y} = -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$(1)=(2) \Rightarrow \boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}}$, $\boxed{\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}}$

Cachy-Rimean condition

$\Rightarrow f(z)$是解析函數

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ex： $f(z)=z^3$

$f(z)=z^3=(x+iy)^3=\underbrace{x^3-3xy^2}_{=u}+i\underbrace{(3x^2y-y^3)}_{=v}$

① $\frac{\partial u}{\partial x}=3x^2-3y^2$

$\frac{\partial v}{\partial y}=3x^2-3y^2$

② $\frac{\partial v}{\partial x}=-6xy=-\frac{\partial u}{\partial y}$

$\therefore f(z)=z^3$是解析函數

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entire function：在複數平面上所有點都是解析函數

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ex： $f(z)=z^*=\bar{z}=x-iy$

$u=x$ ， $v=-y$

$\frac{\partial u}{\partial x}=1$ ， $\frac{\partial v}{\partial y}=-1$

$\Rightarrow \therefore f(z)=\bar{z}$不是 analytic Function

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ex： $\nabla^2 u=0$ $\qquad$ $\nabla^2 v=0$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ $\qquad$ $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

u：harmonic function $\qquad$ u,v 互為共軛（conjugate）

v：harmonic function

$f(z)=u+iv$ ， $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ ， $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})+\frac{\partial}{\partial y}(-\frac{\partial u}{\partial x})=0$

若 $f(z)$是解析函數 $\Rightarrow$ u,v 都是 harmonic function

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ex： $u=x^2-y^2-y$是 harmonic function $v=$ ？

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x \;\Rightarrow\; v=2xy+g(x)$$

$$\frac{\partial u}{\partial y}=-2y-1 \text{，} -\frac{\partial v}{\partial x}=-2y-g'(x)$$

$$\therefore\; g'(x)=1$$

$$\therefore\; g(x)=x+\alpha$$

$$\Rightarrow\; v=2xy+x+\alpha$$

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ex： $f(z)=u+iv$ ， $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ ， $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \Rightarrow \frac{df}{dz^*}=0$

$$z=x+iy \qquad x=\frac{1}{2}(z+z^*)$$

$$z^*=x-iy \qquad y=\frac{1}{2i}(z-z^*)$$

$$\frac{\partial}{\partial z}=\frac{\partial x}{\partial z}\frac{\partial}{\partial x}+\frac{\partial y}{\partial z}\frac{\partial}{\partial y}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial z^*}=\frac{\partial x}{\partial z^*}\frac{\partial}{\partial x}+\frac{\partial y}{\partial z^*}\frac{\partial}{\partial y}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$$

$$f=u+iv$$

$$\frac{df}{dz^*}=\frac{1}{2}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(u+iv)$$

$$=\frac{1}{2}[\underbrace{(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y})}_{=0}+i\underbrace{(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})}_{=0}]=0$$

Cauchy’s Integral Theorem

$f(z)$在 D 上爲解析函數， $\oint f(z)dz=0$

D：simple connected domain

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ex： $\oint_c e^z dz=0$

$\because$ e^z 是解析函數，又在封閉迴路面作積分$\therefore$爲 0

Stoke’s Theorem

$$\int(\nabla\times\vec{v})\cdot d\vec{a}=\oint\vec{v}\cdot d\vec{l}$$

z 分量 $\Rightarrow$ $\int(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})dxdy=\oint(v_1dx+v_2dy)$

pf： $\oint f(z)dz = \oint (u+iv)(dx+idy)$

$$= \oint (udx - vdy) + i\oint (vdx + udy)$$

$$\oint (udx - vdy) = \int (\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y})dxdy = 0$$

$u = v_1$ ， $-v = v_2$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\oint (vdx + udy) = \int (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy = 0$$

$v = v_1$ ， $u = v_2$

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ex： $\oint \cos z dz = 0$

$\oint z^n dz = 0$

$\oint \sec z dz = \oint_c \frac{1}{\cos z} = 0$

要考慮所選的路徑

應用數學

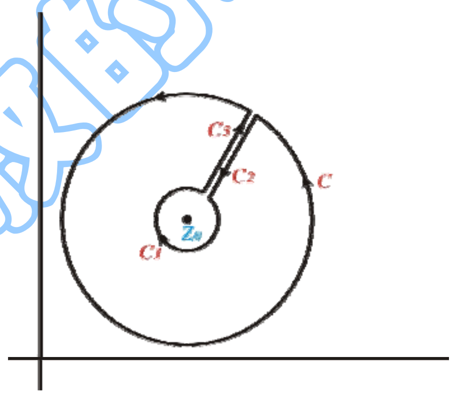
Cauchy's Integral Formula

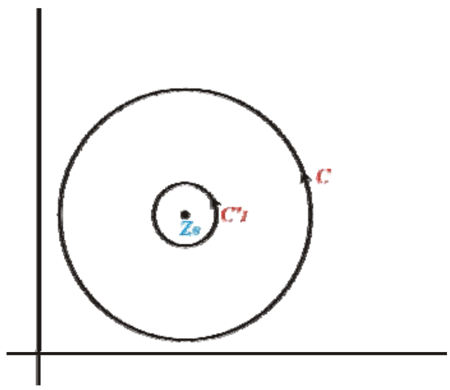
$$\oint \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

$f(z)$爲解析函數， z_0 ：residue

取（counterclockwise）逆時針

$\frac{f(z)}{z - z_0}$ mermorphic function





<pf>： $\oint_{c+c_1+c_2+c_3} \frac{f(z)}{z - z_0} dz = 0$

$$\because \oint_{c_2+c_3} \frac{f(z)}{z-z_0}dz = 0$$

$$\therefore \int_c \frac{f(z)}{z-z_0}dz + \int_{c'} \frac{f(z)}{z-z_0}dz = 0$$

$$\Rightarrow \oint_c \frac{f(z)}{z-z_0}dz = \int_{c_0'} \frac{f(z)}{z-z_0}dz$$

$$z = re^{i\theta} + z_0 \text{ ， } dz = rie^{i\theta}d\theta$$

$$\Rightarrow \lim_{x\to 0}\oint_{c_1'} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}(rie^{i\theta}d\theta) = if(z_0)\oint d\theta$$

$$= 2\pi i f(z_0)$$

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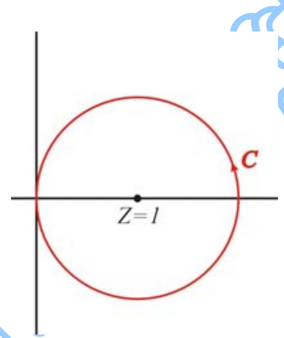
ex：$\oint \frac{e^z}{z-1}dz$ （ $z=1$ 爲 pole 極點）

$$\oint \frac{e^z}{z-1}dz = 2\pi i f(1)$$

$$= 2\pi i e$$



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ex：$\oint \frac{\sin z}{z^2+1}dz$ ，c：$|z|=2$　　　$z=\pm i$ 爲極點

$$\oint \frac{\sin z}{z^2+1}dz = \oint \frac{\sin z}{(z-i)(z+i)}dz$$

$$= 2\pi i\frac{\sin i}{(2i)} + 2\pi i\frac{\sin(-i)}{(-2i)}$$

$$= 2\pi \sin i \text{ 或 } 2\pi i \sinh 1$$

應用數學

Derivatives of analytic function

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$\oint_c$ ：simple connected domain 上的封閉曲線

$f(z_0)$ 微分 n 次 $f^{(n)}(z)$ 存在， $f(z)$ 爲 analysis function

<pf>： $f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}[\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-(z_0+\Delta z)}dz - \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}dz]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}[\frac{1}{2\pi i}\oint \frac{\Delta z f(z)}{(z-z_0-\Delta z)(z-z_0)}dz]$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}dz$$

同理 $f''(z_0) = \frac{2!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^3}dz$

$$f'''(z_0) = \frac{3!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^4}dz$$

$$\vdots$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz$$

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ex： $\oint \frac{\cos z}{(z-\pi i)^2}dz$

$$\oint \frac{\cos z}{(z-\pi i)^2}dz = \frac{2\pi i}{1!}(\cos z)'\Big|_{z=\pi i}$$

$$= 2\pi i(-\sin z)\Big|_{\pi i} \qquad (\sin \pi i = i\sinh \pi)$$

$$= 2\pi \sinh \pi$$

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ex： $\oint \frac{e^{2z}}{(z+1)^4}dz$ ，c： $|z|=3$（如果未包含 pole 的話，積分爲 0，例如：c： $|z|=\frac{1}{2}$ ）

$$\oint \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3}(e^{2z})'''\Big|_{z=-1}$$

$$= \frac{8\pi i}{3}e^{-2}$$

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ex： $\oint \frac{z^4-3z^2+6}{(z+i)^3}dz$ ，c： $|z|=2$

$$\oint \frac{z^4-3z^2+6}{(z+i)^3}dz = \frac{2\pi i}{2!}(z^4-3z^2+6)''\Big|_{z=-i}$$

$$= \pi i(12z^2-6)\Big|_{z=-i}$$

$$= \pi i(-12-6)$$

$$= -18\pi i$$

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ex： $\oint \frac{e^z}{(z-1)^4(z^2+4)}dz$ ，c： $|z|=\frac{3}{2}$

$$=\frac{2\pi i}{3!}\left(\frac{e^z}{z^2+4}\right)'''\Big|_{z=1}+2\pi i\,\cancel{\frac{e^z}{(z-1)^4(z-2i)}}^{=0}\Big|_{z=-2i}+2\pi i\,\cancel{\frac{e^z}{(z-1)^4(z+2i)}}^{=0}\Big|_{z=2i}$$

∵ $|z|=\frac{3}{2}$ ，if $|z|=5$ 就要考慮！

$$=\frac{6\pi e}{25}i$$

Taylor Series

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\ldots$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$f(z)$ 以 z_0 爲圓心的圓 C 內，爲解析函數。

<pf>： $f(z)=\frac{1}{2\pi i}\oint\frac{f(z^*)}{z^*-z}dz^*$

$$\frac{1}{z^*-z}=\frac{1}{(z^*-z_0)-(z-z_0)}$$

$$=\frac{1}{(z^*-z_0)(1-\frac{z-z_0}{z^*-z_0})}$$

$$=\frac{1}{(z^*-z_0)}\left(1+\frac{z-z_0}{z^*-z_0}+\frac{(z-z_0)^2}{(z^*-z_0)^2}+\ldots\right)$$

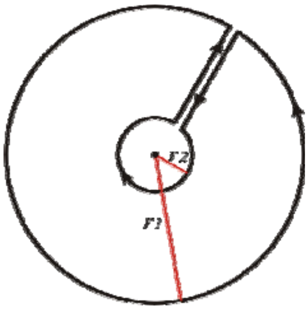
$$=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z^*-z_0)^{n+1}}$$

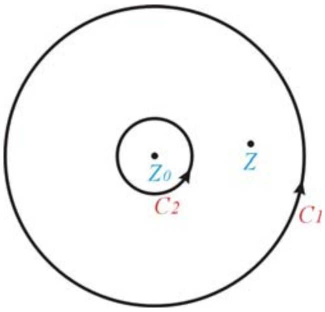
$$f(z)=\frac{1}{2\pi i}\oint f(z^*)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z^*-z_0)^{n+1}}dz^*$$

$$=\sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint\frac{f(z^*)dz^*}{(z^*-z_0)^{n+1}}\right](z-z_0)^n$$

Laurent's Series

設 z_0 爲 $f(z)$ 的孤立奇異點（isolated singularity），而 $f(z)$ 在以 z_0 爲中心， r_1 及 r_2 爲半徑之兩圓周中間環形區域 R 中爲解析函數，則在 R 中每一點函數值可表示成 $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$





$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z^*)}{z^*-z}dz^*$$

$$=\underbrace{\frac{1}{2\pi i}\oint_{c1}\frac{f(z^*)}{z^*-z}dz^*}_{(1)}-\underbrace{\frac{1}{2\pi i}\oint_{c2}\frac{f(z^*)}{z^*-z}dz^*}_{(2)}$$ （若 c2 為順時針則改為+）

第一項：$\frac{1}{2\pi i}\oint_{c1}\frac{f(z^*)}{z^*-z}dz^*=\sum_{n=0}^{\infty}a_n(z-z_0)^n$

$$a_n=\frac{1}{2\pi i}\oint_{c1}\frac{f(z^*)}{z^*-z_0}dz^*$$

第二項：$-\frac{1}{2\pi i}\oint_{c2}\frac{f(z^*)}{z^*-z}dz^*=-\frac{1}{2\pi i}\oint_{c2}\frac{f(z^*)}{(z^*-z_0)-(z-z_0)}dz^*$

（$\because |z-z_0|>|z^*-z_0|$，$z^*$在 c2 上）

$$=\frac{1}{2\pi i}\oint_{c2}\frac{f(z^*)}{(z-z_0)(1-\frac{z^*-z_0}{z-z_0})}dz^*$$

$$=\frac{1}{2\pi i}\oint_{c2}\frac{f(z^*)}{(z-z_0)}[1+\frac{z^*-z_0}{z-z_0}+\frac{(z^*-z_0)^2}{(z-z_0)^2}+\ldots]dz^*$$

$$=\sum_{n=0}^{\infty}b_n(z-z_0)^{-n}$$

$$b_n=\frac{1}{2\pi i}\oint_{c2}(z^*-z_0)^{n-1}f(z^*)dz^*$$

令 $b_n=-a_n \Rightarrow f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$

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ex：$z^{-5}\sin z$

$$z^{-5}\sin z=z^{-5}(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots)$$

$$=\underbrace{\frac{1}{z^4}}_{\text{leading term}}-\frac{1}{3!}\frac{1}{z^2}+\frac{1}{5!}-\frac{1}{7!}z^2+\ldots$$

ex： $f(z)=\dfrac{5}{(z+2)(z-3)}$ 在 $z=3$ 的 Laurent's series

pole： $z_0=-2$ ， $z_0=3$

在 $z=3 \Rightarrow f(z)=\dfrac{5}{(z-3)}\dfrac{1}{(z-3)+5}$

$$=\frac{\not{5}}{(z-3)}\cdot\frac{1}{\not{5}(1+\frac{z-3}{5})}$$

$$=\frac{1}{(z-3)}[1-\frac{z-3}{5}+(\frac{z-3}{5})^2-(\frac{z-3}{5})^3+\ldots]$$

$$=\frac{1}{z-3}-\frac{1}{5}+\frac{1}{25}(z-3)-\frac{1}{125}(z-3)^2+\ldots$$

在 $z=-2 \Rightarrow f(z)=\dfrac{5}{(z+2)}\cdot\dfrac{1}{(z+2)-5}$

$$=\frac{\not{5}}{(z+2)}\cdot\frac{1}{(\not{-5})(1-\frac{z+2}{5})}$$

$$=-\frac{1}{(z+2)}[1+\frac{z+2}{5}+(\frac{z+2}{5})^2+(\frac{z+2}{5})^3+\ldots]$$

$$=-\frac{1}{z+2}-\frac{1}{5}-\frac{1}{25}(z+2)-\frac{1}{125}(z+2)^2-\ldots$$

應用數學

isolated singularities

1、pole

$\lim_{z\to z_0}(z-z_0)^m f(z_0)=c\neq 0$ z_0 ：isolated singularity

m 階的極點（pole）

2、essential（本質） singularity

$\lim_{z\to z_0}(z-z_0)^m f(z)$ 不存在

3、removable singularity

若 $\lim_{z\to z_0} f(z)$ 存在

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ex： $\dfrac{\sin z}{z}$

$z=0 \Rightarrow \lim_{z\to 0}\dfrac{\sin z}{z}=1$

$\Rightarrow$ removable singularity

ex： $f(z)=\dfrac{1}{z(z-2)^5}+\dfrac{3}{(z-2)^3}$

$z=0 \Rightarrow$ a simple pole （一階的 pole）

$z=2 \Rightarrow$ 5 階極點

ex： $e^{\frac{1}{z}}$

$e^{\frac{1}{z}}=1+\dfrac{1}{z}+\dfrac{1}{2!}(\dfrac{1}{z})^2+\ldots$

$z=0 \Rightarrow$ essential singularity

應用數學