Residue（留數）

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)+\frac{a_{-1}}{z-z_0}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-3}}{(z-z_0)^3}+\ldots$$

$f(z)$ ：meromorphic function

$\boxed{\oint_c f(z)dz=2\pi i a_{-1}}$ residue theorem

$$\oint_c f(z)dz=\sum_{n=0}^{\infty}a_n\oint(z-z_0)^n dz+a_{-1}\oint\frac{1}{z-z_0}dz+a_{-m}\oint\frac{1}{(z-z_0)^{-m}}dz \quad m=2,3,\ldots$$

$$\oint\underbrace{(z-z_0)^n}_{解析函數}dz=0$$

$$\oint_c\frac{dz}{z-z_0}=\oint_{c1}\frac{dz}{z-z_0}=\lim_{r\to0}\int_0^{2\pi}\frac{rie^{i\theta}d\theta}{re^{i\theta}}=2\pi i$$

$$z=z_0+re^{i\theta}$$

$$dz=rie^{i\theta}d\theta$$

$$\oint\frac{dz}{(z-z_0)^m}=\lim_{r\to0}\int_0^{2\pi}\frac{rie^{i\theta}d\theta}{(re^{i\theta})^m}=\lim_{r\to0}\frac{i}{r^{m-1}}\underbrace{\int_0^{2\pi}e^{i(1-m)\theta}d\theta}_{=0}=0$$

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ex： $f(z)=z^{-4}\sin z$ ， $a_{-1}(0)=?$

$$f(z)=\frac{1}{z^4}(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots)$$

$$=z^3-\frac{1}{3!z}+\frac{z}{5!}-\ldots$$

$$a_{-1}(0)=-\frac{1}{3!}=-\frac{1}{6}$$

$$\oint_c\frac{\sin z}{z^4}dz=2\pi i(-\frac{1}{6})=-\frac{\pi i}{3}$$

或是 $\oint_c\frac{\sin z}{z^4}dz=\frac{2\pi i}{3!}(\sin z)'''\Big|_{z=0}=-\frac{\pi i}{3}$

$$\oint\frac{f(z)}{(z-z_0)^{n+1}}dz=\oint\frac{f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\ldots+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\ldots}{(z-z_0)^{n+1}}dz$$

$$=2\pi i a_{-1}(z_0)=\frac{2\pi i}{n!}f^{(n)}(z_0)$$

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ex：$f(z)=\dfrac{z(z-2)}{(z+1)^2(z^2+4)}$ 之留數

$$a_{-1}(-1)=\frac{d}{dz}\left(\frac{z(z-2)}{z^2+4}\right)\Big|_{z=-1}=-\frac{14}{25}$$

$$a_{-1}(2i)=\frac{z(z-2)}{(z+1)^2(z+2i)}\Big|_{z=2i}=\frac{7+i}{25}$$

∵$(z-2i)$ 僅一次方

$$a_{-1}(-2i)=\frac{z(z-2)}{(z+1)^2(z-2i)}\Big|_{z=-2i}=\frac{7-i}{25}$$

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ex：求 $f(z)=\dfrac{1}{z-\sin z}$ 在 $z=0$ 的留數

$$f(z)=\frac{1}{z-(z-\frac{z^3}{3!}+\frac{z^5}{5!}-...)}=\frac{1}{\frac{z^3}{3!}(1-\underbrace{\frac{3!}{5!}z^2+...}_{=x})}$$

$$\frac{1}{1-x}=1+x+x^2+x^3+...$$

$$=\frac{3!}{z^3}[1+(\frac{3!}{5!}z^2+...)+(\frac{3!}{5!}z^2+...)^2+...]$$

$$a_{-1}(0)=\frac{3!3!}{5!}=\frac{3}{10}$$

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ex：$\oint_c \dfrac{e^{2z}}{(z+1)^4}dz$，c：$|z|=3$

$$\oint_c \frac{e^{2z}}{(z+1)^4}dz=2\pi i a_{-1}(-1)$$

$$=2\pi i\frac{1}{3!}(e^{2z})''\Big|_{z=-1}$$

$$e^{2z}=...+\frac{(e^{2z})''}{3!}\Big|_{z=-1}(z+1)^3+...$$

$$=\frac{8\pi i}{3e^2}$$

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ex：$\oint_c \dfrac{\tan z}{z^2+\frac{\pi^2}{4}}dz$，c：$|z|=2$

$$\oint_c \frac{\tan z}{z^2+\frac{\pi^2}{4}}dz = 2\pi i[a_{-1}(\frac{\pi}{2}i)+a_{-1}(-\frac{\pi}{2}i)+a_{-1}(\frac{\pi}{2})+a_{-1}(-\frac{\pi}{2})]$$

（$\frac{\pi}{2}$ 、$-\frac{\pi}{2}$ 爲 $\tan z$ 的根）

$$a_{-1}(\frac{\pi}{2}i)=\left.\frac{\tan z}{z+\frac{\pi}{2}i}\right|_{z=\frac{\pi}{2}i} = \frac{\tan\frac{\pi}{2}i}{\pi i} = -\frac{\tanh\frac{\pi}{2}}{\pi}$$

$$a_{-1}(-\frac{\pi}{2}i)=\left.\frac{\tan z}{z-\frac{\pi}{2}i}\right|_{z=-\frac{\pi}{2}i} = \frac{\tan(-\frac{\pi}{2}i)}{-\pi i} = \frac{\tanh\frac{\pi}{2}}{\pi}$$

$\sin(i\theta)=i\sinh\theta$ ，$\cos(i\theta)=\cosh\theta$

$\tan(i\theta)=i\tan\theta$

$$a_{-1}(\frac{\pi}{2})=\lim_{z\to\frac{\pi}{2}}(z-\frac{\pi}{2})\frac{\frac{\sin z}{\cos z}}{z^2+\frac{\pi^2}{4}} = \lim_{z\to\frac{\pi}{2}}\frac{(z-\frac{\pi}{2})}{\cos z}\frac{\sin z}{\frac{\pi^2}{2}} = -\frac{2}{\pi^2}$$

$$a_{-1}(-\frac{\pi}{2})=\lim_{z\to-\frac{\pi}{2}}\frac{(z+\frac{\pi}{2})}{\cos z}\frac{\sin z}{\frac{\pi^2}{2}} = -\frac{2}{\pi^2}$$

$$\oint_c \frac{\tan z}{z^2+\frac{\pi^2}{4}}dz = 2\pi i[\frac{2}{\pi}\tanh\frac{\pi}{2}-\frac{4}{\pi^2}]$$

---

ex：$\oint \frac{e^z}{\cosh z}dz$ ，c：$|z|=5$

$$\boxed{\begin{aligned}&\cosh z=0 \Rightarrow \frac{e^z+e^{-z}}{2}=0 \Rightarrow e^z+e^{-z}=0 \Rightarrow e^{2z}+1=0\\ &\Rightarrow e^{2z}=-1=e^{i(\pi+2n\pi)} \Rightarrow z=\frac{i}{2}(2n+1)\pi\end{aligned}}$$

$$\oint \frac{e^z}{\cosh z}dz = 2\pi i[a_{-1}(\frac{\pi}{2}i)+a_{-1}(-\frac{\pi}{2}i)+a_{-1}(\frac{3\pi}{2}i)+a_{-1}(-\frac{3\pi}{2}i)]$$

$$a_{-1}(\frac{\pi}{2}i)=\lim_{z\to\frac{\pi}{2}i}(z-\frac{\pi}{2}i)\frac{e^z}{\cosh z} = \lim_{z\to\frac{\pi}{2}i}(\frac{z-\frac{\pi}{2}i}{\cosh z})e^{\frac{\pi}{2}i} = (\frac{1}{\sinh\frac{\pi}{2}i})e^{\frac{\pi}{2}i} = \frac{i}{i}=1$$

$$\sinh\frac{\pi}{2}i=\frac{e^{\frac{\pi}{2}i}-e^{\frac{\pi}{2}i}}{2}=\frac{2i}{2}$$

$$e^{\frac{\pi}{2}i}=\cancel{\cos\frac{\pi}{2}}^{=0}+i\sin\frac{\pi}{2}$$

$$a_{-1}(-\frac{\pi}{2}i)=1$$

$$a_{-1}(\frac{3\pi}{2}i)=1$$

$$a_{-1}(-\frac{3\pi}{2}i)=1$$

$$\oint\frac{e^z}{\cosh z}dz=8\pi i$$

（I）三角函數定積分

$$I=\int_0^{2\pi}F(\sin\theta,\cos\theta)d\theta$$

$$\cos\theta=\frac{1}{2}(z+\frac{1}{z})$$

$$\sin\theta=\frac{1}{2i}(z-\frac{1}{z})$$

$$z=e^{i\theta}\ \Rightarrow\ dz=ie^{i\theta}d\theta\ \Rightarrow\ d\theta=\frac{dz}{iz}$$

---

ex： $I=\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}$ ，c：$|z|=1$

$$=\oint_c\frac{\frac{dz}{iz}}{\sqrt{2}-\frac{1}{2}(z+\frac{1}{z})}=\frac{2}{i}\oint_c\frac{dz}{z[2\sqrt{2}-(z+\frac{1}{z})]}=2i\oint_c\frac{dz}{z^2-2\sqrt{2}z+1}$$

$$=2i\oint_c\frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

$$=2i\oint_c(\cancel{\frac{\frac{1}{2}}{z-\sqrt{2}-1}}^{=0}+\frac{-\frac{1}{2}}{z-\sqrt{2}+1})dz$$

$$z^2-2\sqrt{2}z+1=0$$

$$z=\frac{2\sqrt{2}\pm\sqrt{8-4}}{2}=\sqrt{2}\pm1\text{（正不合）}$$

$$\because\sqrt{2}+1>1$$

---

ex： $I=\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta$

$$z=e^{i\theta}\ \Rightarrow\ \cos\theta=\frac{z+z^{-1}}{2}\ \Rightarrow\ d\theta=\frac{dz}{iz}$$

$$z^3=e^{i3\theta}\ ，\ \cos 3\theta=\frac{z^3+z^{-3}}{2}$$

$$I=\oint_c\frac{\frac{1}{2}(z+z^{-1})}{5-4\cdot\frac{z^3+z^{-3}}{2}}\frac{dz}{iz}=-\frac{1}{4i}\oint_c\frac{z^6+1}{z^3(z-2)(z-\frac{1}{2})}dz$$

z 前面的係數盡量爲 1，不然後續處理會很麻煩！

$$=-\frac{1}{4i}\cdot 2\pi i[\frac{21}{4}-\frac{65}{12}]=\frac{\pi}{12}$$

---

ex： $I=\int_0^{\pi}\frac{d\theta}{1+\sin^2\theta}$

$$=\int_0^{\pi}\frac{d\theta}{1+\frac{1-\cos 2\theta}{2}}=\int_0^{\pi}\frac{2d\theta}{3-\cos 2\theta}\underset{\substack{d\theta=\frac{1}{2}d\phi\\ 2\theta=\phi}}{=}\int_0^{2\pi}\frac{d\phi}{3-\cos\phi}$$

$$=\oint_c\frac{2i}{z^2-6z+1}dz=2\pi i(-\frac{i}{\sqrt{8}})=\frac{\pi}{\sqrt{2}}$$

---

ex： $I=\int_0^{2\pi}\frac{\sin^2\theta}{2+\cos\theta}d\theta$

$$=\frac{i}{2}\oint\frac{z^4-2z^2+1}{z^2(z^2+4z-1)}dz=2\pi i\cdot\frac{i}{2}(-4+2\sqrt{3})=\pi(4-2\sqrt{3})$$

---

ex： $I=\int_0^{2\pi}\frac{d\theta}{1-2p\cos\theta+p^2}$ ， $0<p<1$

$$=\oint_c\frac{\frac{dz}{iz}}{1-2p(\frac{z+z^{-1}}{2})+p^2}=\oint_c\frac{i}{pz^2+(1+p^2)z+p}dz$$

$$=\oint_c\frac{i}{(z-p)(pz+1)}dz=2\pi i(\frac{i}{p^2-1})=\frac{2\pi}{1-p^2}$$

應用數學

（II） $\int_{-\infty}^{\infty}f(x)dx$

ex： $I=\int_0^\infty \frac{1}{1+x^2}dx=\tan^{-1}x\Big|_0^\infty=\frac{\pi}{2}$

（補圖）

另解： $I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

$J=\oint\frac{1}{1+z^2}dz=2\pi i a_{-1}(i)$

$\boxed{1}=\int_0^\infty\frac{dr}{1+r^2}=\int_0^\infty\frac{dx}{1+x^2}$

$\boxed{2}=\int_0^\infty\frac{dz}{1+z^2}=\int_\infty^0\frac{-dr}{1+(-r)^2}=\int_{-\infty}^0\frac{dx}{1+x^2}$ （ $r\to -x$ ）

（ $z=re^{i\pi}=-r$ ）

$\boxed{3}=\int_0^\infty\frac{dz}{1+z^2}=\lim_{r\to\infty}\int_0^\pi\frac{rie^{i\theta}d\theta}{1+r^2e^{2i\theta}}\ge\lim_{r\to\infty}\int_0^\pi\frac{re^{i\theta}d\theta}{1+r^2}\to 0$

$J=\int_{-\infty}^\infty\frac{dx}{1+x^2}+\cancelto{=0}{\int_{C_R}\frac{dz}{1+z^2}}=2I$

$a_{-1}(i)=(z-i)\frac{1}{(z-i)(z+i)}\Big|_{z=i}=\frac{1}{2i}$

$I=\frac{1}{2}\cdot 2\pi i\cdot(\frac{1}{2i})=\frac{\pi}{2}$

---

ex： $I=\int_{-\infty}^\infty\frac{1}{1+x^4}dx$

$z^4+1=0$

$z^4=-1=e^{i(\pi+2n\pi)}$

$\therefore\ z=e^{i\frac{\pi}{4}},e^{i\frac{3\pi}{4}},e^{i\frac{5\pi}{4}},e^{i\frac{7\pi}{4}}$

（補圖）

$J=\oint\frac{1}{1+x^4}dx$

$=2\pi i\left[a_{-1}\left(e^{i\frac{\pi}{4}}\right)+a_{-1}\left(e^{i\frac{3\pi}{4}}\right)\right]$

$$=\int_{-\infty}^{\infty}\frac{dx}{1+x^4}+\int_{CR}\overset{=0}{\cancel{\frac{dz}{1+z^4}}}=I$$

$$a_{-1}\left(e^{i\frac{\pi}{4}}\right)=\left(z-e^{i\frac{\pi}{4}}\right)\frac{1}{z^4+1}\bigg|_{z=e^{i\frac{\pi}{4}}}=\frac{1}{4z^3}\bigg|_{z=e^{i\frac{\pi}{4}}}=\frac{1}{4}e^{-i\frac{3\pi}{4}}$$

$$a_{-1}\left(e^{i\frac{3\pi}{4}}\right)=\left(z-e^{i\frac{3\pi}{4}}\right)\frac{1}{z^4+1}\bigg|_{z=e^{i\frac{3\pi}{4}}}=\frac{1}{4z^3}\bigg|_{z=e^{i\frac{3\pi}{4}}}=\frac{1}{4}e^{-i\frac{9\pi}{4}}=\frac{1}{4}e^{-i\frac{\pi}{4}}$$

$$I=2\pi i\left(\frac{1}{4}e^{-i\frac{3\pi}{4}}+\frac{1}{4}e^{-i\frac{\pi}{4}}\right)$$

$$=\frac{\pi i}{2}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}+\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$$

$$=\frac{\pi i}{2}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=\frac{\pi}{\sqrt{2}}$$

---

ex： $I=\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)^2}$

$z^2+1=0$， $z=\pm i$

$z^2+4=0$， $z=\pm 2i$

$$J=\oint\frac{dz}{(z^2+1)(z^2+4)^2}$$

$$=\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)^2}+\int_{CR}\frac{dz}{(z^2+1)(z^2+4)^2}$$

$$J=2\pi i\left[a_{-1}(i)+a_{-1}(2i)\right]$$

$$a_{-1}(i)=\frac{1}{(z^2+1)(z^2+1)^2}\bigg|_{z=i}=\frac{1}{(2i)(3^2)}=\frac{1}{18i}=\frac{-i}{18}$$

$$a_{-1}(2i)=\frac{d}{dz}\frac{1}{(z^2+1)(z^2+1)^2}\bigg|_{z=2i}=\frac{11i}{288}$$

$$J=2\pi i\left(-\frac{i}{18}+\frac{11i}{288}\right)=\frac{5\pi}{144}$$

---

ex： $I=\int_{-\infty}^{\infty}\frac{x^2-1}{x^4+5x^2+4}dx$

$$J=\oint\frac{z^2-1}{z^4+5z^2+4}=I$$

$$=2\pi i\left[a_{-1}(i)+a_{-1}(2i)\right]$$

$$=2\pi i\left[\frac{i}{3}-\frac{5}{12}i\right]$$

$z^4+5z^2+4=(z^2+4)(z^2+1)$

$z=\pm 2i,\pm i$

$$=\frac{\pi}{6}$$

---

ex： $I=\int_0^\infty \frac{dx}{x^4+x^2+1}$

$$=\frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^4+x^2+1}$$

$$J=\oint \frac{dz}{z^4+z^2+1}$$

$$z^4+z^2+1=0$$

$$z^2=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}=\begin{cases} e^{i\frac{2\pi}{3}} \\ e^{i\frac{4\pi}{3}} \end{cases}$$

$$=\int_{-\infty}^\infty \frac{dx}{x^4+x^2+1}+\int_{CR}\frac{dz}{z^4+z^2+1}=2I$$

$$=2\pi i\left[a_{-1}\left(e^{i\frac{\pi}{3}}\right)+a\left(e^{i\frac{2\pi}{3}}\right)\right]$$

$$I=\frac{\sqrt{3}}{6}\pi$$

---

ex： $I=\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ $\qquad a>0$

$$=\frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^2}$$

$$J=\oint \frac{dz}{(z^2+a^2)^2}=\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^2}+\int_{CR}\frac{dz}{(z^2+a^2)^2}=2I$$

$$=2\pi i\left[a_{-1}(ai)\right]$$

$$a_{-1}(ai)=\frac{d}{dz}\left[(z+ai)^{-2}\right]\Big|_{z=ai}=-2(z+ai)^{-3}\Big|_{z=ai}$$

$$=-2\frac{1}{(2ai)^3}=-\frac{i}{4a^3}$$

$$I=\frac{1}{2}\cdot 2\pi i\cdot\left(\frac{-i}{4a^3}\right)=\frac{\pi}{4a^3}$$

應用數學

（III） $\int_{-\infty}^\infty e^{iax}f(x)dx$

$a>0$ 取上半平面

$a<0$ 取下半平面

Jordan’s lemma

$a>0$

$$I_R=\int e^{iaz}\underset{\lim_{|z|\to\infty} f(z)\to 0}{f(z)}\ dz$$

$$0<\arg z<\pi\ ,\ \ z=Re^{i\theta}\ ,\ \ dz=Rie^{i\theta}d\theta$$

$$=\int_0^{\pi} e^{ia(Re^{i\theta})} f(Re^{i\theta})Rie^{i\theta}d\theta$$

$$\lim_{R\to\infty} f(Re^{i\theta})=0\ \Rightarrow\ \lim_{R\to\infty}\left|f(Re^{i\theta})\right|<\varepsilon$$ （無限小的數）

$$\left|e^{i\theta}\right|=1,\ \ \left|\int f(z)dz\right|\le\int\left|f(z)\right|dz$$

$$=\left|I_R\right|$$

$$=\int_0^{\pi}\left|e^{ia|R\cos\theta+i\sin\theta|}\right|\left|f(Re^{i\theta})\right|\left|Rie^{i\theta}\right|d\theta$$

$$\le 2\varepsilon R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta$$

$$\underset{\theta\to(\pi-\theta)}{\underline{\int_{\frac{\pi}{2}}^{\pi} e^{-aR\sin\theta}d\theta}}=\int_{\frac{\pi}{2}}^{0} e^{-aR\sin\theta}(-d\theta)$$

$$\le 2\varepsilon R\int_0^{\frac{\pi}{2}} e^{-aR\frac{2}{\pi}\theta}d\theta$$

$$=2\varepsilon R\cdot\frac{1-e^{-aR}}{\frac{aR^2}{\pi}}$$

$$\xrightarrow{R\to\infty}\frac{\pi}{a}\varepsilon$$

$$\therefore\ \ \lim_{R\to\infty}\left|I_R\right|\to 0$$

（補圖）

$$\Rightarrow\ \sin\theta\ge\frac{2}{\pi}\theta$$

---

ex：$I=\int_{-\infty}^{\infty}\frac{e^{ix}}{1+x^2}dx$

$$J=\oint\frac{e^{iz}}{1+z^2}dz=I+\int_{CR}\frac{e^{iz}}{1+z^2}dz=2\pi i a_{-1}(i)$$

$(z\to x+iy)$

$$a_{-1}(i)=\left.\frac{e^{iz}}{z+i}\right|_{z=i}=\frac{e^{-1}}{2i}$$

$$\therefore\ \ I=2\pi i\left(\frac{1}{2ie}\right)=\frac{\pi}{e}$$

---

ex：$I=\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx$

$$J = I_m \oint \frac{ze^{i\pi z}}{z^2+2z+5}dz \qquad z=-1\pm 2i$$

$$= I_m\left[2\pi i a_{-1}\left(-1+2i\right)\right] = I$$

$$a_{-1}(-1+2i) = \left.\frac{\pi e^{i\pi z}}{z+1+2i}\right|_{z=-1+2i}$$

$$= \frac{(1-2i)e^{-2\pi}}{4i} \qquad e^{-i\pi} = \cos(-\pi)+i\sin(-\pi) = -1$$

$$\Rightarrow I = \underset{\text{取虛部}}{\underline{\text{Im}}}\left[2\pi i\frac{(1-2i)e^{-2\pi}}{4i}\right] = -\pi e^{-2\pi}$$

---

ex： $I = \int_0^\infty \frac{\cos mx}{1+x^2}dx$ $\qquad m<0$（取下半面）

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\text{Re}(e^{imx})}{1+x^2}dx$$

$$= \frac{1}{2}\text{Re}\int_{-\infty}^{\infty} \frac{e^{imx}}{1+x^2}dx$$

$J = \oint \frac{e^{imz}}{1+z^2}dz = -2\pi i a_{-1}(i)$ 下半面呈現順時針，所以有負號。

$$a_{-1}(i) = \left.\frac{e^{imz}}{z-i}\right|_{z=-i} = \frac{e^m}{-2i}$$

$$I = \frac{1}{2}\text{Re}\left(-2\pi i\cdot\frac{e^m}{-2i}\right) = \frac{1}{2}\pi e^m$$

---

ex： $I = \int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$ $\qquad (a>0, b>0, a\neq b)$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$$

$$= \frac{1}{2}\text{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx$$

$J = \oint \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz \qquad z=\pm ai, \pm bi$ $\qquad$ iz 係數爲正，取上半部

$$= 2\pi i\left[a_{-1}(ai)+a_{-1}(bi)\right] = I_0$$

$$a_{-1}(ai) = \left.\frac{e^{iz}}{(z+ai)(z^2+b^2)}\right|_{z=ai} = \frac{e^{-a}}{2ai(-a^2+b^2)}$$

$$a_{-1}(bi) = \frac{e^{-b}}{2bi(a^2-b^2)}$$

$$I=\frac{1}{2}\operatorname{Re}I_0=\frac{1}{2}\frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

若 $a=b \Rightarrow I=\frac{\pi}{4a^3}(a+1)e^{-a}$

（IV）通過極點的積分

ex： $I=\int_{-\infty}^{\infty}\frac{1}{x-1}dx$

（補圖）

$$z=1+re^{i\theta}$$
$$dz=rie^{i\theta}d\theta$$

$$J=\oint\frac{1}{z-1}dz$$
$$=\underset{\text{principle value}}{\underline{P.V.}}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{x-1}}_{I}+\lim_{r\to0}\underbrace{\int_{\pi}^{0}\frac{rie^{i\theta}d\theta}{1+re^{i\theta}-1}}_{-\pi i}=0$$
$$\Rightarrow\ I-\pi i=0\ \Rightarrow\ I=\pi i$$

If

（補圖）

$$J=P.V.\int_{-\infty}^{\infty}\frac{dx}{x+1}+\int_{\pi}^{2\pi}\frac{rie^{i\theta}d\theta}{1+re^{i\theta}-1}$$
$$=2\pi ia_{-1}(i)=2\pi i$$
$$\Rightarrow\ I+\pi i=2\pi i\ \Rightarrow\ I=\pi i$$

∴ 不管有無包含 pole 均會相同。

---

ex： $I=P.V.\int_{0}^{\infty}\frac{\sin x}{x}dx$

$$= \frac{1}{2} P.V.\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$$

$$= \frac{1}{2} P.V.\,\mathrm{Im}\underbrace{\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx}_{I_0}$$

（補圖）

$$z = re^{i\theta}$$

$$dz = rie^{i\theta}d\theta$$

$$J = \oint\frac{e^{iz}}{z}dz = \underbrace{P.V.\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx}_{I_0} + \underbrace{\lim_{r\to 0}\int_{\pi}^{0}\frac{e^{i(re^{i\theta})}}{re^{i\theta}}rie^{i\theta}d\theta}_{-\pi i} = 0$$

$$\Rightarrow \quad I_0 - \pi i = 0 \quad \Rightarrow \quad I_0 = \pi i \quad \Rightarrow \quad I = \frac{1}{2}\mathrm{Im}\, I_0 = \frac{\pi}{2}$$

---

ex： $I = \int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{1}{4}\int_{-\infty}^{\infty}\frac{1-\cos 2x}{x^2}dx = \frac{1}{4}\mathrm{Re}\int_{-\infty}^{\infty}\frac{1-e^{i2x}}{x^2}dx$$

$$J = \oint\frac{1-e^{i2z}}{z^2}dz = I_0 + \left[-\pi i a_{-1}(0)\right]$$

$$= I_0 + \lim_{r\to 0}\int_{\pi}^{0}\frac{1-e^{i2(re^{i\theta})}}{r^2e^{2i\theta}}rie^{i\theta}d\theta$$

$$= I_0 + \int_{\pi}^{0}\frac{-2ie^{i\theta}}{e^{i\theta}}id\theta$$

$$= I_0 + (-2i)(-\pi)(i) = 0$$

$$I_0 = 2\pi \quad \Rightarrow \quad I = \frac{\pi}{2}$$

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ex： $I = \int_{-\infty}^{\infty}\frac{1}{(x-1)(x^2+3)}dx$

$$J = \oint\frac{1}{(z-1)(z^2+3)}dz = 2\pi i a_{-1}(\sqrt{3}i)$$

$$I + \lim_{r\to 0}\int_{\pi}^{0}\frac{ire^{i\theta}d\theta}{(re^{i\theta})\left[\left(1+re^{i\theta}\right)^2+3\right]} = I + \left(-\frac{i\pi}{4}\right)$$

$$a_{-1}(\sqrt{3}i) = \left.\frac{1}{(z-1)(z+\sqrt{3}i)}\right|_{\sqrt{3}i} = \frac{1}{(\sqrt{3}i-1)(2\sqrt{3}i)}$$

$$2\pi i\cdot\frac{1}{(\sqrt{3}i-1)(2\sqrt{3}i)} = I - \frac{i\pi}{4}$$

$$= \frac{\pi(-\sqrt{3}i-1)}{4\sqrt{3}} = -\frac{\pi}{4}i - \frac{\pi}{4\sqrt{3}}$$

$$\Rightarrow \quad I = -\frac{\pi\sqrt{3}}{12}$$

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ex： $I = \int_{-\infty}^{\infty} \frac{1}{(x+1)(x^2+4)} dx$

（補圖）

$$z = -1 + re^{i\theta}$$
$$dz = rie^{i\theta} d\theta$$

$$J = \oint \frac{1}{(z+1)(z^2+4)} dz = 2\pi i a_{-1}(2i)$$

$$= I + \lim_{r\to 0} \int_{\pi}^{0} \frac{ire^{i\theta} d\theta}{re^{i\theta}\left[\left(-1-re^{i\theta}\right)^2 + 4\right]} = I - \frac{i\pi}{5}$$

$$a_{-1}(2i) = \left.\frac{1}{(z+1)(z+2i)}\right|_{2i} = \frac{1}{(2i+1)(4i)}$$

$$2\pi i \cdot \frac{1}{(2i+1)(4i)} = I - \frac{i\pi}{5}$$

$$\Rightarrow \quad I = \frac{\pi}{10}$$

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ex： $I = \int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx$ （ $a > 0$ ）

$$= \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2} dx$$

（補圖）

$$J = \oint \frac{e^{iz}}{a^2 - z^2} dz = I_0 + I_1 + I_2 = 0$$

$$I_1 = -\lim_{r\to 0} \int \frac{e^{iz} dz}{(z+a)(z-a)} = -\lim_{r\to 0} \int_{\pi}^{0} \frac{rie^{i\theta} d\theta \cdot e^{i(-a+re^{i\theta})}}{re^{i\theta}\left[-2a + re^{i\theta}\right]} = \frac{\pi i}{-2a} e^{-ia}$$

$$I_2 = -\lim_{r\to 0} \int \frac{e^{iz} dz}{(z+a)(z-a)} = -\lim_{r\to 0} \int_{\pi}^{0} \frac{rie^{i\theta} d\theta \cdot e^{i(a+re^{i\theta})}}{re^{i\theta}\left[2a + re^{i\theta}\right]} = -\frac{(-\pi i)}{2a} e^{ia}$$

$$I_0 + \left(-\frac{\pi i}{2a}e^{-ia}\right) + \left(\frac{\pi i}{2a}e^{ia}\right) = 0$$

$$I_0 - \frac{\pi i}{2a}\left(e^{-ia} - e^{ia}\right) = 0$$

$$I_0 - \frac{\pi i}{2a}\left(-2i\sin a\right) = 0$$

$$\Rightarrow\ I_0 = \frac{\pi \sin a}{a} = I$$

三角函數定積分

考慮 $\int_0^{2\pi} f(\cos\theta,\sin\theta)\cdot d\theta$ 的積分值

在 $C:|z|=1$ 之圓上，任一點之參數式爲 $z=e^{i\theta}$ ，$\frac{1}{z}=e^{-i\theta}$

則 $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(z+z^{-1})$

$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}(z-z^{-1})$

$dz = ie^{i\theta}d\theta$ ，$d\theta = \frac{1}{i}e^{-i\theta}dz = -iz^{-1}dz$

則 $\int_0^{2\pi} f(\cos\theta,\sin\theta)\cdot d\theta = \oint_c f\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$, $C:|z|=1$

在利用留數（Residue）定理來求積分值。

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ex：求 $\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$

在 $C:|z|=1$ 之圓上， $z=e^{i\theta}$ ， $d\theta=\frac{1}{iz}dz$

$\cos\theta=\frac{1}{2}(z+z^{-1})$ ，$\sin\theta=\frac{1}{2i}(z-z^{-1})$

$$\therefore \int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_c \frac{\frac{1}{iz}dz}{3-(z+\frac{1}{z})+\frac{1}{2i}(z-\frac{1}{z})}$$

$$= \oint_c \frac{2dz}{(1-2i)z^2+6iz-(2i+1)}$$

$$= 2\pi i\cdot Res\left(\frac{-i}{1-2i}\right)$$

奇異點 $z=\frac{-5i}{1-2i},\frac{-i}{1-2i}$

$$Res\left(\frac{-i}{1-2i}\right) = \lim_{z\to\frac{-i}{1-2i}}\left(z-\frac{-i}{1-2i}\right)\left(\frac{2}{(1-2i)z^2+6iz-(1+2i)}\right) = \frac{1}{2i}$$

$$\therefore \text{ 原式} = 2\pi i \cdot Res\left(\frac{-i}{1-2i}\right) = 2\pi i \cdot \frac{1}{2i} = \pi$$

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ex：求 $\int_0^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$， $a<1$

$$\int_0^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$$

在 $C:|z|=1$ ， $z=e^{i\theta}$ ， $d\theta=\frac{1}{iz}dz$ ， $\cos\theta=\frac{1}{2}(z+z^{-1})$

$$\int_0^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$$

$$= \frac{1}{2}\oint_c \frac{\frac{dz}{iz}}{1-a\left(z+\frac{1}{z}\right)+a^2}$$

$$= \frac{-1}{2i}\oint_c \frac{dz}{az^2-(a^2+1)z+a}$$

$$= \frac{-1}{2i}\oint_c \frac{dz}{(az-1)(z-a)}$$

$$= \frac{-1}{2i}\cdot 2\pi i \cdot Res(a)$$

$$Res(a) = \lim_{z\to a}(z-a)\frac{1}{(az-1)(z-a)} = \frac{1}{a^2-1}$$

$$\therefore \int_0^{\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = -\pi\cdot\frac{1}{a^2-1} = \frac{\pi}{1-a^2}$$

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ex：求 $\int_0^{2\pi} \frac{\cos 2\theta}{13-5\cos\theta}d\theta$

在 $C:|z|=1$ ， $z=e^{i\theta}$ ， $\cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2} = \frac{1}{2}(z^2+z^{-2})$

$d\theta=\frac{1}{iz}dz$ ， $\cos\theta=\frac{1}{2}(z+z^{-1})$

$$\Rightarrow \int_0^{2\pi} \frac{\cos 2\theta}{13-5\cos\theta}d\theta = \oint_c \frac{\frac{1}{2}(z^2+z^{-2})}{13-5\left(\frac{z+z^{-1}}{2}\right)}\frac{dz}{iz}$$

$$= \oint_c \frac{1}{-iz^2}\cdot\frac{z^4+1}{5z^2-26z+5}dz$$

$$= \oint_c \frac{1}{-iz^2}\cdot\frac{z^4+1}{(5z-1)(z-5)}dz$$

$$= 2\pi i\left[Res(0)+Res(\frac{1}{5})\right]$$

$$Res(0)=\lim_{z\to 0}\frac{d}{dz}\left(z^2\cdot\frac{z^4+1}{(-iz^2)(5z-1)(z-5)}\right)$$

$$=\lim_{z\to 0}\left(\frac{1}{-i}\right)\frac{d}{dz}(z^4+1)(5z-1)^{-1}(z-5)^{-1}$$

$$\boxed{\begin{aligned}&\frac{d}{dz}(z^4+1)(5z-1)^{-1}(z-5)^{-1}\\&=(4z^3)(5z-1)^{-1}(z-5)^{-1}+(z^4+1)(5z-1)^{-2}(5)(z-5)^{-1}\\&\qquad\qquad+(z^4+1)(5z-1)^{-1}(z-5)^{-2}\\&=\frac{4z^3(5z-1)(z-5)-(z^4+1)\left[5(z-5)+(5z-1)\right]}{(5z-1)^2(z-5)^2}\end{aligned}}$$

$$=\lim_{z\to 0}\frac{4z^3(5z-1)(z-5)-(z^4+1)\left[5(z-5)+(5z-1)\right]}{-i(5z-1)^2(z-5)^2}$$

$$=\frac{-26}{25i}$$

$$Res(\frac{1}{5})=\lim_{z\to\frac{1}{5}}\left(z-\frac{1}{5}\right)\cdot\frac{-(z^4+1)}{(iz^2)(5z-1)(z-5)}$$

$$=\lim_{z\to\frac{1}{5}}\cdot\frac{-(z^4+1)}{(i5z^2)(z-5)}$$

$$=\frac{-\left[\left(\frac{1}{5}\right)^4+1\right]}{i5\cdot\left(\frac{1}{5}\right)^2\left(\frac{1}{5}-5\right)}$$

$$=\frac{313}{300i}$$

$$\therefore\int_0^{2\pi}\frac{\cos 2\theta}{13-5\cos\theta}d\theta=2\pi i\left(-\frac{26}{25i}+\frac{313}{300i}\right)=\frac{\pi}{150}$$

有理函數瑕積分

………未完待續！