PART I : Vectors

§1.1. Vectors in two and three dimensions

△ Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$: Algebraic notations

A vector in $\mathbb{R}^2$: (a_1, a_2) [eg. $(1, 2)$ or $(\pi, \sqrt{17})$]

" $\mathbb{R}^3$: (a_1, a_2, a_3) [eg. $(\pi, \sqrt{2}, \sin(5))$

~ an ordered (有序的) pair or triple of real numbers

~ Single real numbers = scalars 純量

~ notations : $\vec{A}$ or $\mathbb{A} = (a_1, a_2, a_3)$

$$\boxed{\vec{A} = (a_1, a_2),\ \vec{B} = (b_1, b_2),\quad \vec{A} = \vec{B} \Longleftrightarrow a_1 = b_1 \text{ and } a_2 = b_2}$$

Ex. $\vec{a} = (1, 2)$, $\vec{b} = (3/3, 6/3) \Rightarrow \vec{a} = \vec{b}$

$\vec{c} = (1, 2, 3)$, $\vec{d} = (2, 1, 3) \Rightarrow \vec{c} \neq \vec{d}$

△ Vector addition (加法) and scalar multiplication (乘法)

$$\boxed{\begin{array}{l} \vec{a} = (a_1, a_2, a_3),\ \vec{b} = (b_1, b_2, b_3) \\ \text{The sum (和) of } \vec{a} \text{ and } \vec{b} \text{ is } \vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \end{array}}$$

~ $(0, 1, 3) + (\sqrt{17}, \pi, -1) = (\sqrt{17}, 1 + \pi, 2)$

~ Properties : 1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

3. Null or zero vector : $\vec{a} + \vec{0} = \vec{a}$ for any $\vec{a}$

$\vec{0} = (0, 0, 0)$ in $\mathbb{R}^3$

$$\boxed{\begin{array}{l} \vec{a} = (a_1, a_2, a_3) \text{ is a vector in } \mathbb{R}^3,\ k \in \mathbb{R} \text{ is a scalar,} \\ k\vec{a} = (ka_1, ka_2, ka_3) \end{array}}$$

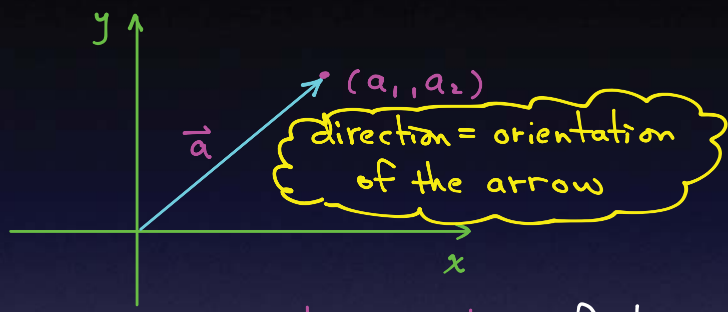
~ $k = 7$, $\vec{a} = (2, 0, \sqrt{2}) \Rightarrow k\vec{a} = (14, 0, 7\sqrt{2})$

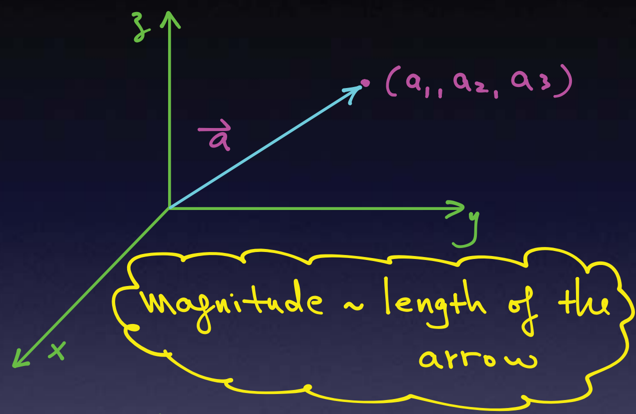
~ Properties : $k, \ell \in \mathbb{R}$, 1. $(k + \ell)\vec{a} = k\vec{a} + \ell\vec{a}$

2. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, 3. $k(\ell\vec{a}) = (k\ell)\vec{a} = \ell(k\vec{a})$

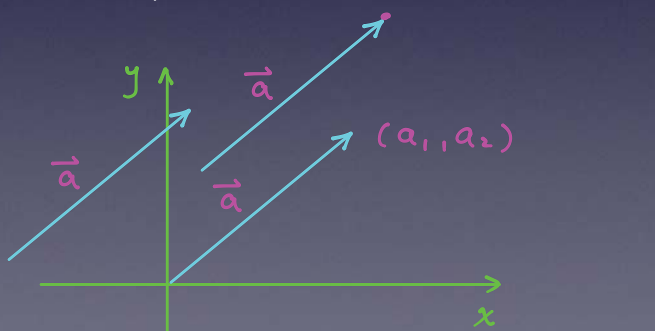
△ Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$: The geometric (几何) notation

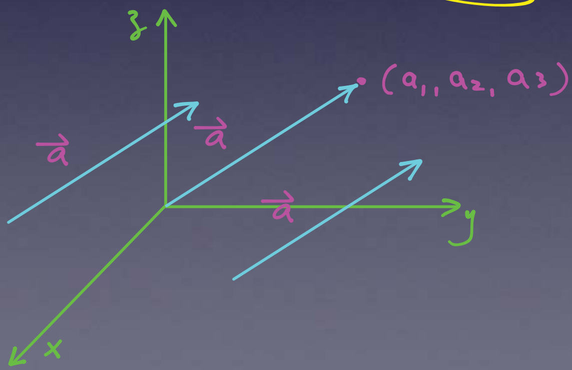
A vector in $\mathbb{R}^2$ ($\mathbb{R}^3$) corresponds to a point in $\mathbb{R}^2$ ($\mathbb{R}^3$)





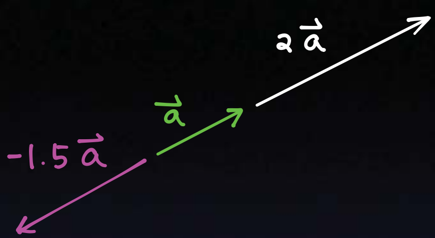
The position vector of the point (a_1, a_2)

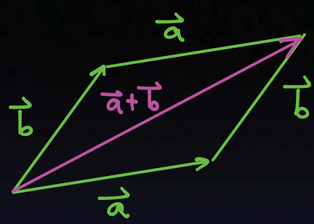


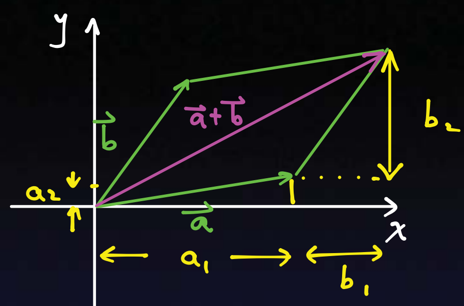


Parallel (平行) translated position vector

○ The relation between the geometric and algebraic notions:







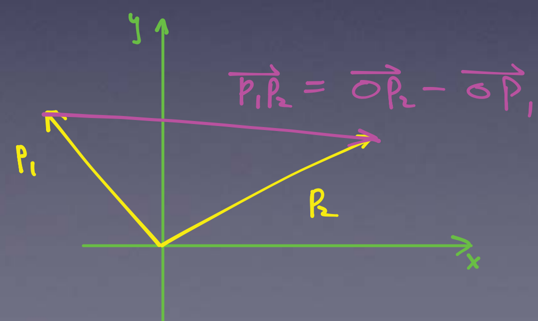
$$-\vec{b} \equiv -1 \cdot \vec{b}, \quad \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

○ Displacement vectors (位移向量) vs. Position (位置) vectors

Given two points $P_1 : (x_1, y_1, z_1)$, $P_2 : (x_2, y_2, z_2)$ in $\mathbb{R}^3$

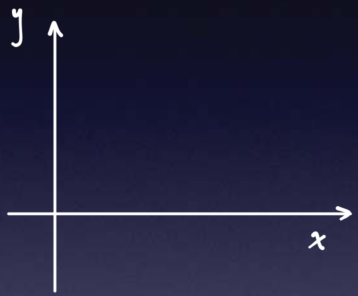
The displacement vector from P_1 to P_2 is

$$\overrightarrow{P_1P_2} \equiv (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$



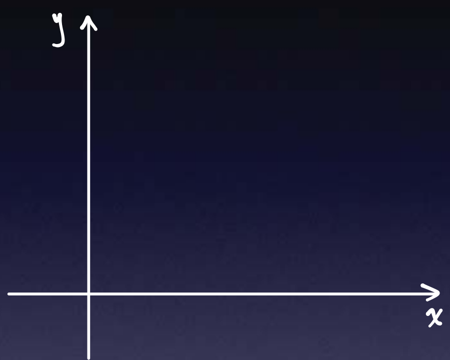
△ More examples and exercises:

1. Graph the vectors, $\vec{a} = (3, 2)$, $b = (-1, 1)$,
   Also calculate and graph $\vec{a} + \vec{b}$, $\frac{1}{2}\vec{a}$, $\vec{a} + 2\vec{b}$
   What are the magnitudes of the above vectors?



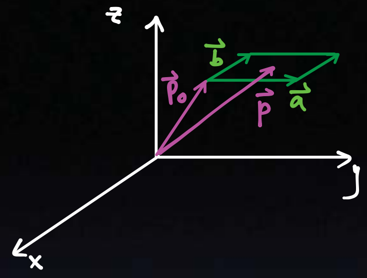
2. If $(-12, 9, z) + (x, 7, -3) = (2, y, 5)$, what are x, y, and z?

3. (a) Let $\vec{a} = (2, 0)$ and $\vec{b} = (1, 1)$. For $0 \leq s \leq 1$ and $0 \leq t \leq 1$, consider the vector $\vec{x} = s\vec{a} + t\vec{b}$. Explain why the vector $\vec{x}$ lies in the parallelogram (平行四边形) determined by $\vec{a}$ and $\vec{b}$.



(b) Now suppose that $\vec{a} = (2, 2, 1)$ and $\vec{b} = (0, 3, 2)$
   Describe the set of vectors $\{ \vec{x} = s\vec{a} + t\vec{b},\ 0 \leq s \leq 1,\ 0 \leq t \leq 1 \}$

(c)



Point $P_0 : (x_0, y_0, z_0)$

$\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

由 $\vec{a}, \vec{b}$ 所圍平行四邊形範圍內之點 $P:(x,y,z)$ 該如何以 $P_0, \vec{a}, \vec{b}$ 描述?

4. Given two forces: $\vec{F_1} = (2, 7, -1)$ and $\vec{F_2} = (3, -2, 5)$, act on an object,

(a) What is the resultant (or) net force? (合力)

(b) Suppose there is a third force F_3, such that there is no net force, $\vec{F_3} = ?$

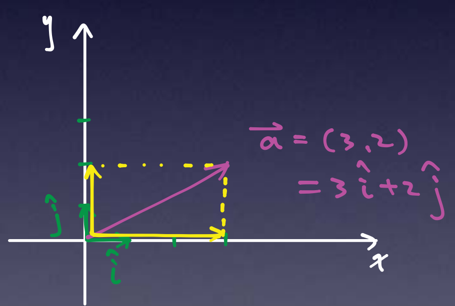
§ 1.2. More about vectors:

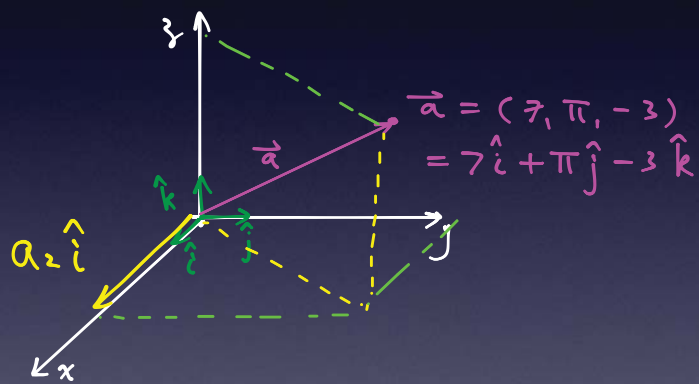
□ The standard basic vectors (基底向量)

In $\mathbb{R}^2$, $\hat{i} \equiv (1, 0)$, $\hat{j} \equiv (0, 1)$. $\vec{a} = (a_1, a_2)$

$$\Rightarrow \vec{a} = (a_1, 0) + (0, a_2) = a_1 (1, 0) + a_2 (0, 1)$$

$$= a_1 \hat{i} + a_2 \hat{j}$$





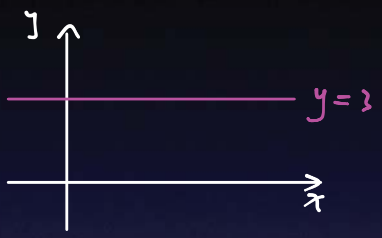
In $\mathbb{R}^3$, $\hat{i} \equiv (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$

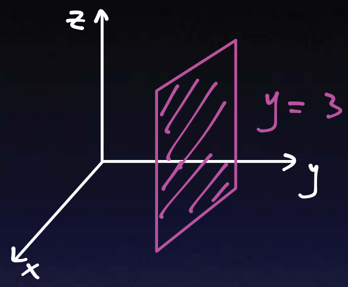
$\vec{a} = (a_1, a_2, a_3) = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

a_1, a_2, a_3 are the vector components (分量) along the x-, y-, and z- axes.

□ Parametric equations of lines (直線的參數方程式)

In $\mathbb{R}^2$, $y=mx+b$ or $Ax+By=c$ describes straight lines. Is this still true in $\mathbb{R}^3$?



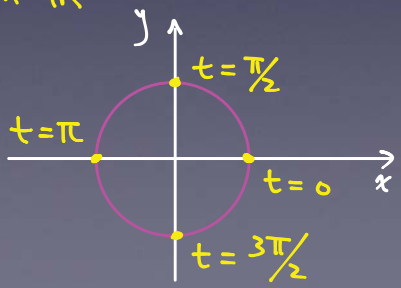


A curve : a one-dimensional object ~ a function of one variable (變數)

The points on the curve (x,y) : $\begin{cases} x=f(t) \\ y=g(t) \end{cases}$, t : a real parameter (variable)

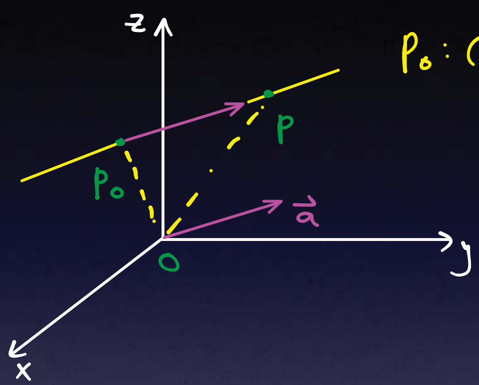
in $\mathbb{R}^2$

~ a parametric equation



$$x^2+y^2=4 \sim \begin{cases} x=2\cos t \\ y=2\sin t \end{cases}, \quad 0 \le t < 2\pi$$

~ a curve in $\mathbb{R}^3$: $\begin{cases} x=f(t) \\ y=g(t) \\ z=h(t) \end{cases}$



$P_0 : (x_0, y_0, z_0)$, $\vec{a} = (a_1, a_2, a_3)$

$\overrightarrow{OP_0} = (x_0, y_0, z_0)$ ~ position vector

$P : (x,y,z) \in$ the line, $\vec{r} = \overrightarrow{OP} = (x,y,z)$

$$\vec{r} = \overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{a}$$

⇒ The vector parametric equation for the line through the point $P_0 : (b_1, b_2, b_3)$, and parallel to $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is

$$\boxed{\vec{r}(t) = \vec{b} + t\vec{a}}$$

⇒
$$x = a_1 t + b_1$$
$$y = a_2 t + b_2$$
$$z = a_3 t + b_3$$

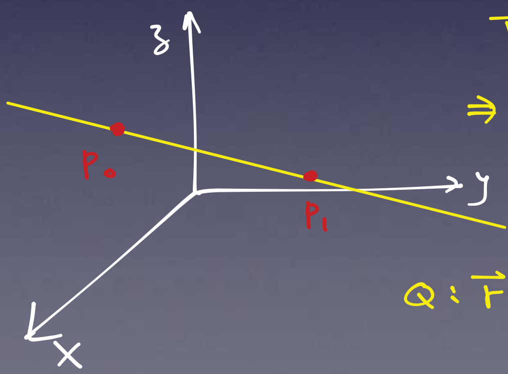
This can be easily generalized to $\mathbb{R}^n$!

Ex. The parametric equation of the line through $(1,-2,3)$ and parallel to the vector $\pi\hat{i} - 3\hat{j} + \hat{k}$

$$\vec{r}(t) = (1,-2,3) + t\,(\pi,-3,1)$$

$$\Rightarrow \begin{cases} x = t\pi + 1 \\ y = -3t - 2 \\ z = t + 3 \end{cases}$$

Ex. The parametric equation of the line through the points $P_0:(1,-2,3)$, $P_1:(0,5,-1)$



$$\vec{r} = \vec{OP_0} + t\,\vec{P_0P_1}, \quad \vec{P_0P_1} = (-1,7,-4)$$

$$\Rightarrow \begin{cases} x = 1 - t \\ y = -2 + 7t \\ z = 3 - 4t \end{cases}$$

Q: $\vec{r} = \vec{OP_1} + u\,\vec{P_0P_1}$ find the relation between the parameters u and t.

☐ The parametric equations for a line (or curve) are never unique (不唯一)

$$\vec{r}(t) = \vec{OP_0} + t\,\vec{P_0P_1} = \vec{OP_1} + u\,\vec{P_0P_1} = \vec{a} + v\,\vec{b}, \quad \vec{b} \,/\!/\, \vec{P_0P_1}$$

The symmetric form of a line (in $\mathbb{R}^3$):

$$\begin{aligned} x &= a_1 + v b_1 \\ y &= a_2 + v b_2 \\ z &= a_3 + v b_3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} v &= \frac{x-a_1}{b_1} \\ v &= \frac{y-a_2}{b_2} \\ v &= \frac{z-a_3}{b_3} \end{aligned} \quad \Rightarrow \quad \boxed{\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}}$$

In the previous example: $\dfrac{x-1}{-1} = \dfrac{y+2}{7} = \dfrac{z-3}{-4}$

Subtracting 1 from each side of the above

$$\Rightarrow \frac{x}{-1} = \frac{y-5}{7} = \frac{z+1}{-4} \sim \text{the same line}$$

Ex. The intersection (交點) of a line and a plane (平面)

The line : $\begin{cases} x = t+5 \\ y = -2t-4 \\ z = 3t+7 \end{cases}$, The plane : $3x+2y-7z=2$

$3(t+5)+2(-2t-4)-7(3t+7)=2$

$\Rightarrow -22t = 2+49+8-15 = 44, \quad t=-2$

$\Rightarrow$ The point is $(3, 0, 1)$

Ex. Determine whether the two lines $\begin{cases} x = t+1 \\ y = 5t+6 \\ z = -2t \end{cases}$ and $\begin{cases} x = 3t-3 \\ y = t \\ z = t+1 \end{cases}$

intersect ?

$\begin{cases} t_1+1 = 3t_2-3 \\ 5t_1+6 = t_2 \\ -2t_1 = t_2+1 \end{cases}$ $\quad -2t_1 = (5t_1+6)+1,$

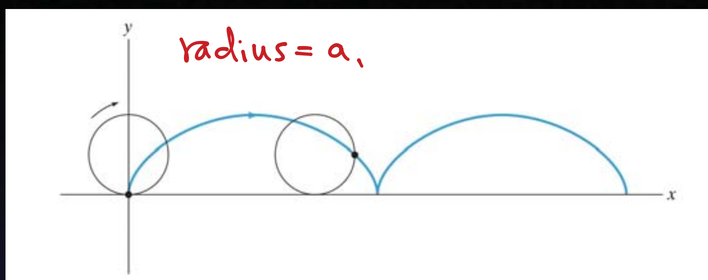
$\Rightarrow 7t_1 = -7, \; t_1=-1, \; t_2 = 1$

代入第(1)式 $(-1)+1 = 3\cdot 1-3$

$\Rightarrow$ 確實相交, 且交點為 $(0, 1, 2)$

□ Parametric equations in general :

Ex A cycloid (擺線) :



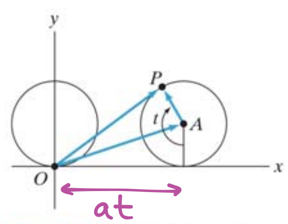


Figure 1.27 The result of the wheel in Figure 1.26 rolling through a central angle of t.

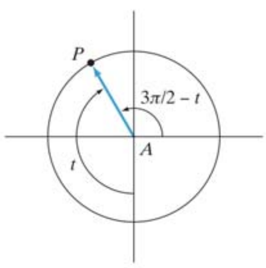


Figure 1.28 $\overrightarrow{AP}$ with its tail at the origin.

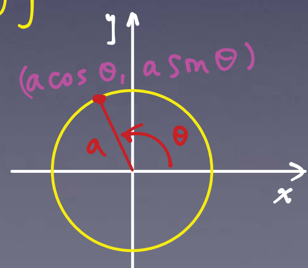
key : P點之運動 = 圓心 A 之平移 + AP 之轉動 (圓)

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\overrightarrow{OA} = at\,\hat{i} + a\,\hat{j}$ (弧長 = $\overset{a}{\text{半徑}}$ × $\overset{t}{\text{角度}}$)

$\overrightarrow{AP} = a\cos\left(\frac{3\pi}{2}-t\right)\hat{i} + a\sin\left(\frac{3\pi}{2}-t\right)\hat{j}$

$= -a\sin t\,\hat{i} - a\cos t\,\hat{j}$



$\Rightarrow \begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$

Exercises :

1. $\vec{a}_1 = (1,1)$, $\vec{a}_2 = (1,-1)$

   (a) $\vec{b} = (3,1) = c_1\vec{a}_1 + c_2\vec{a}_2$. Find c_1, c_2

   (b) Repeat part (a) for a vector $\vec{b} = (3,-5)$

   (c) Show that any vector in $\mathbb{R}^2$ may be written as $c_1\vec{a}_1 + c_2\vec{a}_2$ for appropriate choice of the scalars c_1 and c_2
   (This means $\vec{a}_1$ and $\vec{a}_2$ form a basis vector in $\mathbb{R}^2$)

2. Find the equations for the lines so described :

   a) The line through $(2,-1,5)$ and parallel to vector $\hat{i} + 3\hat{j} - 6\hat{k}$

   b) The line in $\mathbb{R}^3$ through the points $(2,1,2)$ and $(3,-1,5)$

   c) The symmetric form for the line : $x = 5-2t$, $y = 3t+1$, $z = 6t-4$

3. (a) Are the two lines with symmetric form
   $$\frac{x-1}{5} = \frac{y+2}{-3} = \frac{z+1}{4} \quad \text{and} \quad \frac{x-4}{10} = \frac{y-1}{-5} = \frac{z+5}{8}$$
   the same ? why or why not ?

   (b) $\dfrac{x-2}{3} = \dfrac{y-1}{7} = \dfrac{z}{5}$ and $\dfrac{x+a}{-6} = \dfrac{y+b}{-14} = \dfrac{z+5}{-10}$ represent the same lines. Find a, b

4. (a) Find where the line having the symmetric form
   $$\frac{x-3}{6} = \frac{y+2}{3} = \frac{z}{5}$$
   intersects the plane $3x + 3y + z = 22$

   (b) Do the lines l_1: $x = 2t+1$, $y = -3t$, $z = t-1$ and l_2: $x = 3t+1$, $y = t+5$, $z = 7-t$ intersect? Explain your answer.

§ 1.3 The Dot Product

- Two useful concepts of vector product:

  i) The Euclidean inner product or dot product

  ii) The "cross" or vector product (Only for vectors in $\mathbb{R}^3$)

△ The dot product of two vectors

> $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ are vectors in $\mathbb{R}^3$
>
> The dot product of $\vec{a}$ and $\vec{b}$ is $\vec{a} \cdot \vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$

- For vectors in $\mathbb{R}^2$, $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

Ex. $(1, -2, 5) \cdot (2, 1, 3) = 2 - 2 + 15 = 15$

$(3\hat{i} + 2\hat{j} - \hat{k}) \cdot (\hat{i} - 2\hat{k}) = 3 \cdot 1 + 2 \cdot 0 + (-1)(-2) = 5$

- The dot product is also known as the scalar product
- the scalar product of two vectors produces a single real number.

△ Properties of dot product:

$\vec{a}, \vec{b}$ are two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), and $k \in \mathbb{R}$ is any scalar, we have

> i) $\vec{a} \cdot \vec{a} \geq 0$, and $\vec{a} \cdot \vec{a} = 0$ if and only if $\vec{a} = 0$
>
> ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
>
> iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
>
> iv) $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$

Proof of i): Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 \geq 0$

$$a_1^2 + a_2^2 + a_3^2 = 0 \iff a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

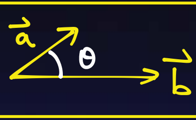
△ Definition:

> The length (or norm, or magnitude) of a vector $\vec{a} = (a_1, a_2, a_3)$
>
> $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ (or $|\vec{a}|$)

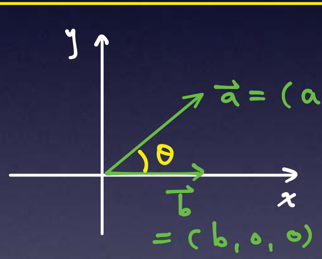
- It is nothing but the length of the arrow
- Obviously, we have $\boxed{\vec{a}\cdot\vec{a} = |\vec{a}|^2}$

- If $\vec{a}$ and $\vec{b}$ are any two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

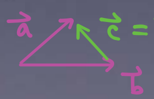


Proof:



$\vec{a}\cdot\vec{b} = ab\cos\theta$ #

△ It is important to realize that the result product is coordinate independent!



$$\underbrace{|\vec{c}|^2}_{\text{coord. indep}} = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \underbrace{|\vec{a}|^2 + |\vec{b}|^2}_{\text{coord. indep.}} - 2\,\vec{a}\cdot\vec{b}$$

$\Rightarrow$ $\vec{a}\cdot\vec{b}$ is coord. indep., too!

~ The law of Cosine: $c^2 = a^2 + b^2 - 2ab\cos\theta$

△ 內積有什麼用？

1. Angle between two vectors:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta = \cos^{-1}\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Ex. $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} - \hat{k}$

$$\Rightarrow \vec{a}\cdot\vec{b} = 1\cdot 0 + 1\cdot 1 + 0\cdot(-1) = 1,\quad |\vec{a}| = \sqrt{1^2+1^2} = \sqrt{2},\quad |\vec{b}| = \sqrt{2}$$

$$\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

- $\cos\theta = 0 \Leftrightarrow \vec{a}\cdot\vec{b} = 0 \quad (|\vec{a}|, |\vec{b}| \neq 0)$

> Vectors $\vec{a}$ and $\vec{b}$ are perpendicular (垂直) or orthogonal (正交) when $\vec{a}\cdot\vec{b} = 0$

Ex: The vector $\hat{i}+\hat{j}$ and the vector $\hat{i}-\hat{j}+\hat{k}$ are orthogonal:

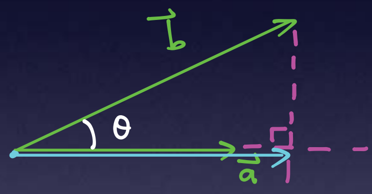
$$(\hat{i}+\hat{j})\cdot(\hat{i}-\hat{j}+\hat{k}) = 1 - 1 + 0 = 0$$

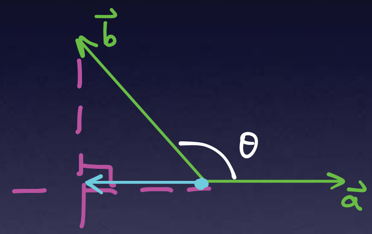
- Note that by definition, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \rightarrow$ 正交基底

2. Vector projection (向量投影)

The projection of $\vec{b}$ onto $\vec{a}$ (denoted by $\longrightarrow$ below)





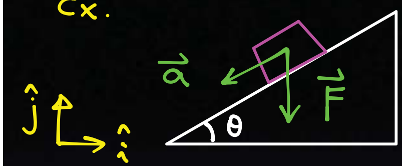
$|k\vec{u}| = k|\vec{u}|$

$$\begin{cases} \text{norm} = |\vec{b}|\cos\theta \\ \text{unit vector } \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} \end{cases} \Rightarrow \text{the projection} = |\vec{b}|\cos\theta\,\frac{\vec{a}}{|\vec{a}|}$$

$$= |\vec{b}|\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\,\frac{\vec{a}}{|\vec{a}|}$$

$$\Rightarrow \boxed{\mathrm{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}}$$

Ex.



Let $\vec{a}$ be a unit vector

$\vec{a} = a_1\hat{i} + a_2\hat{j} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j}$

$\vec{F} = -mg\,\hat{j}$ = gravitational force

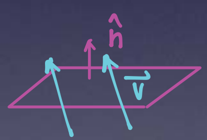
The projection of $\vec{F}$ along the direction of the ramp $\vec{a}$:

$$\mathrm{proj}_{\vec{a}}\vec{F} = \frac{\vec{F}\cdot\vec{a}}{|\vec{a}|^2}\,\vec{a} = +mg\sin\theta\,\vec{a} = -mg\sin\theta(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

For $\theta = 30° = \pi/6$ $\quad \vec{F} = -\frac{1}{4}mg(\sqrt{3}\,\hat{i} + \hat{j})$

$|\mathrm{proj}_{\vec{a}}\vec{F}| = mg\cos\theta$

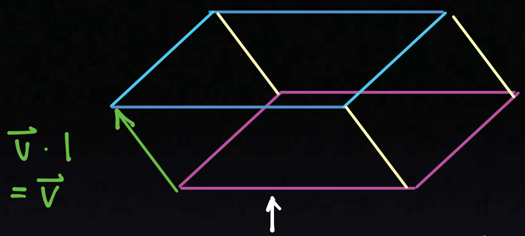
Ex.



A fluid is flowing across a plane with uniform velocity $\hat{n}$

Q: What is the volume of the fluid that passes through the unit area of the plane in unit time?

~ flux (通量)



After one unit time, the fluid passing across the area will have filled the box

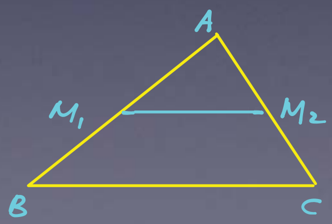
Volume of the box = (area of base) $\cdot$ (height)

$$\text{height} = \left|\,\text{proj}_{\hat{n}}\vec{v}\,\right| = \left|\frac{\vec{v}\cdot\hat{n}}{|\hat{n}|^2}\,\hat{n}\right| = |\vec{v}\cdot\hat{n}| \quad (|k\hat{n}| = k\,|\hat{n}| = k)$$

$\Rightarrow$ $\boxed{\text{the flux} \equiv \vec{v}\cdot\hat{n}}$

△ 利用向量証明一些基本的几何問題：

Ex.



$$\overline{M_1M_2} \parallel \overline{BC} \Rightarrow \frac{\overline{AM_1}}{\overline{AB}} = \frac{\overline{AM_2}}{\overline{AC}} \ (\equiv \alpha)$$

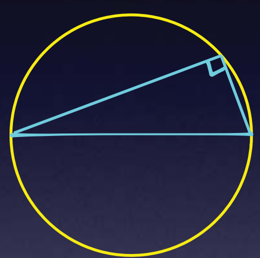
則 $\dfrac{\overline{M_1M_2}}{\overline{BC}} = \alpha$

Proof: $\overrightarrow{AM_1} = \alpha\overrightarrow{AB}$, $\overrightarrow{AM_2} = \alpha\overrightarrow{AC}$

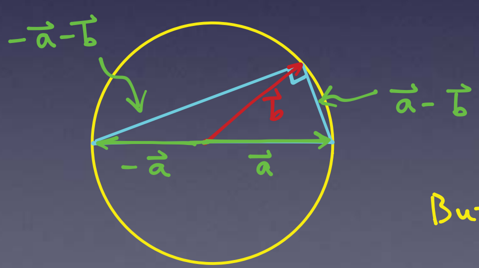
$$\overrightarrow{M_1M_2} = \overrightarrow{AM_2} - \overrightarrow{AM_1} = \alpha(\overrightarrow{AC} - \overrightarrow{AB}) = \alpha\,\overrightarrow{BC}$$

$$\Rightarrow \frac{\overline{M_1M_2}}{\overline{BC}} = \frac{|\overrightarrow{M_1M_2}|}{|\overrightarrow{BC}|} = \alpha$$

Ex



Every angle inscribed in a semicircle is a right angle.



$$(-\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = -(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$$

$$= -(|\vec{a}|^2 - |\vec{b}|^2)$$

But $|\vec{a}| = r = |\vec{b}| \Rightarrow -\vec{a}-\vec{b} \perp \vec{a}-\vec{b}$ #

□ Exercises:

1. Calculate $\vec{a}\cdot\vec{b}$, $|\vec{a}|$, $|\vec{b}|$, the angle between $\vec{a}$ and $\vec{b}$, $\text{proj}_{\vec{a}}\vec{b}$, for vectors listed below:

   a) $\vec{a}=(2,1,0)$, $\vec{b}=(1,-2,3)$

   b) $\vec{a}=\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$, $\vec{b}=2\hat{i}+3\hat{j}-\hat{k}$

2. Give a unit vector that point in the same direction as the vector $2\hat{i}-\hat{j}+\hat{k}$

3. Determine the cosine of the angles between $\vec{a}$ and, respectively the x-, y-, z- axes. $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$

4. a) Show that the vector $|\vec{b}|\vec{a}+|\vec{a}|\vec{b}$ and $|\vec{b}|\vec{a}-|\vec{a}|\vec{b}$ are orthogonal.

   b) Show that $|\vec{b}|\vec{a}+|\vec{a}|\vec{b}$ bisets the angle between $\vec{a}$ and $\vec{b}$.

§ 1.4 The Cross Product:

The cross product of twe vectors in $\mathbb{R}^3$ takes two vectors and produces a third one. $\longrightarrow$ vector product

Definition (定義): (几何定义)

$\vec{a}$ and $\vec{b}$ are two vectors in $\mathbb{R}^3$ (not $\mathbb{R}^2$), and the cross/vector product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$, is the vector whose length and direction are given as follows:

i) length/norm $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

θ is the angle between $\vec{a}$ and $\vec{b}$

ii) direction: right-hand rule $\Rightarrow$ $\boxed{\begin{array}{l}\vec{a}\perp(\vec{a}\times\vec{b}) \text{ and } \vec{b}\perp(\vec{a}\times\vec{b})\\ \vec{a}\parallel\vec{b}\Rightarrow\vec{a}\times\vec{b}=0\end{array}}$

$\vec{0}\times\vec{a}=0$ for any $\vec{a}$

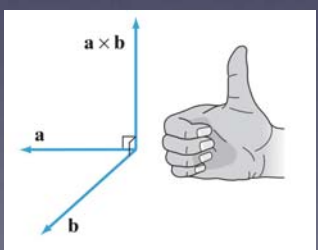
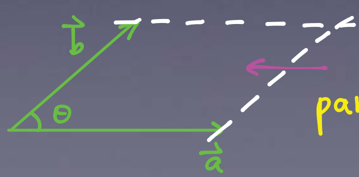
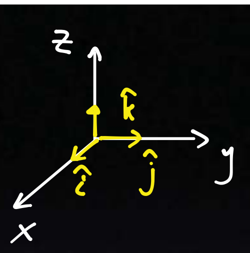


Figure 1.52 The right-hand rule for finding **a** × **b**.



the area of the parallelogram is $|\vec{a}\times\vec{b}|$

Ex.



For the standard basis $\hat{i}, \hat{j}, \hat{k}$, we have

$$\boxed{\hat{i}\times\hat{j}=\hat{k}, \quad \hat{j}\times\hat{k}=\hat{i}, \quad \hat{k}\times\hat{i}=\hat{j}}$$

$$|\hat{i}\times\hat{j}| = |\hat{i}|\,|\hat{j}|\,\sin\pi/2 = 1$$

Properties of the cross product:

$\vec{a}, \vec{b}, \vec{c}$ are vectors in $\mathbb{R}^3$, and $k\in\mathbb{R}$ is a scalar,

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
2. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
3. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$
4. $k(\vec{a}\times\vec{b}) = (k\vec{a})\times\vec{b} = \vec{a}\times(k\vec{b})$

The first is obvious, the rest will be proved later.

Note that i) $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$

ii) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

check: $(\hat{i}\times\hat{j})\times\hat{j} = \hat{k}\times\hat{j} = -\hat{i}$

$\hat{i}\times(\hat{j}\times\hat{j}) = \hat{i}\times\vec{0} = \vec{0}$

iii) $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$$\Rightarrow \vec{a}\times\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

$$= a_1\hat{i}\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \longrightarrow a_1b_2\hat{k}-a_1b_3\hat{j}$$

$$+ a_2\hat{j}\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \longrightarrow -a_2b_1\hat{k}+a_2b_3\hat{i}$$

$$+ a_3\hat{k}\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \longrightarrow a_3b_1\hat{j}-a_3b_2\hat{i}$$

$$\Rightarrow \boxed{\vec{a}\times\vec{b} = \underbrace{(a_2b_3-a_3b_2)}_{(\vec{a}\times\vec{b})_1}\hat{i} + \underbrace{(a_3b_1-a_1b_3)}_{(\vec{a}\times\vec{b})_2}\hat{j} + \underbrace{(a_1b_2-a_2b_1)}_{(\vec{a}\times\vec{b})_3}\hat{k}}$$

可視為 cross product 之代數定义

Ex. $(\hat{i}+3\hat{j}-2\hat{k})\times(2\hat{i}+2\hat{k})$

$$= (3\cdot 2-2\cdot 0)\hat{i} + (-2\cdot 2-1\cdot 2)\hat{j} + (1\cdot 0-3\cdot 2)\hat{k}$$

$$= 6\hat{i}-6\hat{j}-6\hat{k} \quad \#$$

△ Matrices and determinant (矩陣與行列式)

$$M=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}, \quad N=\begin{bmatrix}1 & 3\\ 2 & 7\\ 0 & 0\end{bmatrix}, \quad \begin{array}{cc} & \text{3th column} \\ P= & \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \end{array} \text{2nd row}$$

M_{23} $\qquad$ N_{31} $\qquad$ P_{33}

2×3 (row, column) $\qquad$ 3×2 $\qquad$ 4×4

row $\qquad$ column

△ The determinant (行列式) of a 2×2 and 3×3 matrice.

- $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $\det(A)=\begin{vmatrix}a & b\\ c & d\end{vmatrix}\equiv ad-bc$

- $A=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$, $|A| = a\begin{vmatrix}e & f\\ h & i\end{vmatrix} - b\begin{vmatrix}d & f\\ g & i\end{vmatrix} + c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$

key fact: If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$\boxed{\begin{aligned}\vec{a}\times\vec{b} &= \begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\hat{i} - \begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\hat{j} + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\hat{k}\\ &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\end{aligned}}$$

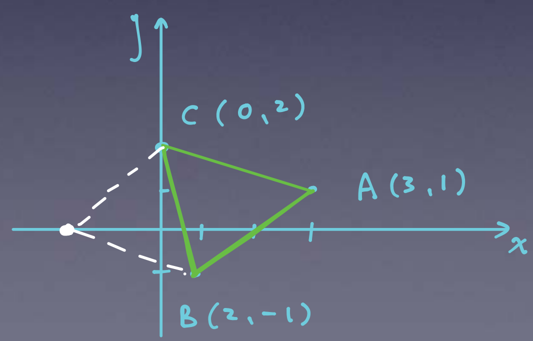
Ex: $(3\hat{i}+2\hat{j}-\hat{k})\times(\hat{i}-\hat{j}+\hat{k})$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & -1\\ 1 & -1 & 1\end{vmatrix} = \begin{vmatrix}2 & -1\\ -1 & 1\end{vmatrix}\hat{i} - \begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix}\hat{j} + \begin{vmatrix}3 & 2\\ 1 & -1\end{vmatrix}\hat{k}$$

$$= \hat{i}-4\hat{j}-5\hat{k}$$

△ Areas and volumes (面積與体積)

The area of the triangle (三角形) ABC:



Area of $\triangle ABC = \frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|$

$\vec{AB} = (-1,-2) = -\hat{i}-2\hat{j}$, $\vec{AC} = (-3,1) = -3\hat{i}+\hat{j}$

$\vec{AB}\times\vec{AC} = (-2\cdot 0 - 0\cdot(-2))\hat{i} + (0\cdot(-3)-(-1)\cdot 0)\hat{j} + (-1\cdot 1-(-2)\cdot(-3))\hat{k}$

$= -7\hat{k}$

$\Rightarrow$ Area of $\triangle ABC = 7/2$ #

Of course, you may use $\frac{1}{2}(\vec{BA}\times\vec{BC})$ to calculate this area.

Ex.

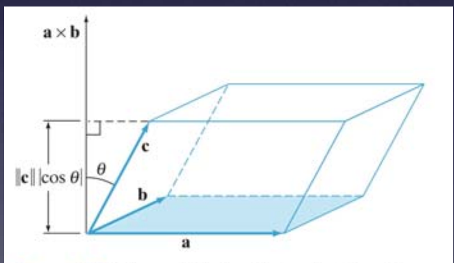


Figure 1.58 The parallelepiped determined by **a**, **b**, and **c**.

The volume of the parallelepiped (平行六面体) determined by the vectors $\vec{a}, \vec{b}, \vec{c}$.

Volume = (area of base) (height)

$= |\vec{a}\times\vec{b}|\,|c|\cos\theta$

$$\boxed{\text{Volume} = \left|\vec{c}\cdot(\vec{a}\times\vec{b})\right|}$$

triple scalar product.

Note that i) $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |(\vec{b}\times\vec{c})\cdot\vec{a}| = |(\vec{c}\times\vec{a})\cdot\vec{b}|$

If fact, if you examine the directions carefully, you may convince yourself that

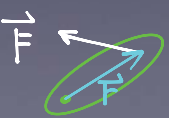
$$\boxed{(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b}}$$

ii) $(\vec{a}\times\vec{b})\cdot\vec{c} = c_1(\vec{a}\times\vec{b})_1 + c_2(\vec{a}\times\vec{b})_2 + c_3(\vec{a}\times\vec{b})_3$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \leftarrow \vec{a}\cdot(\vec{b}\times\vec{c})$$

$\triangle$ Some physical applications:

i) The torque vector (力矩向量)

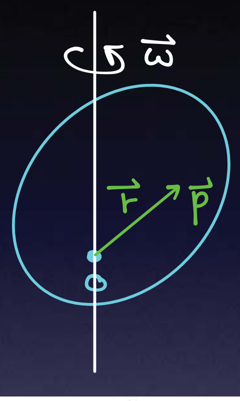


$\vec{\tau} \equiv \vec{r}\times\vec{F}$ ($\vec{\tau} = 0$ if $\vec{r} \parallel \vec{F}$)

$\left(\vec{\tau} = \frac{d\vec{L}}{dt},\ \vec{L} = \vec{r}\times\vec{p}\right)$

ii) Rotation of a rigid body (剛体):

– Angular velocity 角速度:



$\vec r(t) = \vec{OP}$

$\vec v = \lim\limits_{\Delta t \to 0} \frac{\Delta \vec r}{\Delta t}$ the velocity of $\vec P$

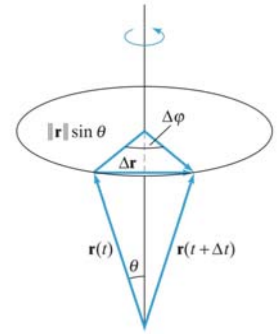
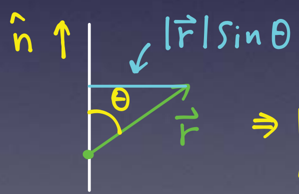


Figure 1.62 A spinning rigid body.

For $\Delta t \to 0$, $|\Delta \vec r| \approx |\vec r| \sin\theta \, \Delta\varphi$



$$\Rightarrow \left| \frac{\Delta \vec r}{\Delta t} \right| \simeq |\vec r| \sin\theta \, \frac{\Delta\varphi}{\Delta t}$$

角度度 $\vec\omega \equiv \lim\limits_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} \hat n$ 轉軸之方向

$\Rightarrow |\vec v| = |\vec\omega \times \vec r|$ or $\boxed{\vec v = \vec\omega \times \vec r}$

Appendix: Proof of $\vec a \times (\vec b + \vec c) = \vec a \times \vec b + \vec a \times \vec c$

i) 用分量代入檢查（以代數定义出発）

$$\begin{aligned}
[\vec a \times (\vec b + \vec c)]_1 &= a_2 (\vec b + \vec c)_3 - a_3 (\vec b + \vec c)_2 \\
&= a_2 (b_3 + c_3) - a_3 (b_2 + c_2) \\
&= (a_2 b_3 - a_3 b_2) + (a_2 c_3 - a_3 c_2) \\
&= (a \times \vec b)_1 + (\vec a \times \vec c)_1
\end{aligned}$$

ii) 以几何定义出発:

$$\begin{aligned}
[\vec a \times (\vec b + \vec c)] \cdot \vec x &= (\vec x \times \vec a) \cdot (\vec b + \vec c) \\
&= (\vec x \times \vec a) \cdot \vec b + (\vec x \times \vec a) \cdot \vec c \\
&= (\vec a \times \vec b) \cdot \vec x + (\vec a \times \vec c) \cdot \vec x \\
&= [(\vec a \times \vec b) + (\vec a \times \vec c)] \cdot \vec x
\end{aligned}$$

This is true for any $\vec x$ in $\mathbb{R}^3 \Rightarrow \vec a \times (\vec b + \vec c) = \vec a \times \vec b + \vec a \times \vec c$

□ Exercises:

1. 計算行列式：$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ -1 & 0 & 3 \end{vmatrix}$

2. 計算 cross product: i) $(3\hat{i}-2\hat{j}+\hat{k})\times(\hat{i}+\hat{j}+\hat{k})$

ii) $(\hat{i}+\hat{j})\times(-3\hat{i}+2\hat{j})$

3. $\vec{a}\times\vec{b}=3\hat{i}-7\hat{j}-2\hat{k}$, $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=?$

4. 四邊形頂點座標為 $(1,2,3)$, $(4,-2,1)$, $(-3,1,0)$, $(0,-3,2)$: i) 請檢查是否為平行四邊形

ii) 計算其面積.

5. 計算頂點為 $(1,0,1)$, $(0,2,3)$, $(-1,5,2)$ 三角形之面積.

6. 求出由向量 $3\hat{i}-\hat{j}$, $-2\hat{i}+\hat{k}$, $\hat{i}-2\hat{j}+4\hat{k}$ 所圍平行六面体 (parallelepiped) 之体積.

7. 对非零向量 $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, 構造出以下物件:

i) A vector orthogonal to $\vec{a}$ and $\vec{b}$

ii) A vector of length 2, orthogonal to $\vec{a}$ and $\vec{b}$

iii) The vector projection of $\vec{b}$ onto $\vec{a}$

iv) A vector with the length of $\vec{b}$ and the direction of $\vec{a}$

v) A vector orthogonal to $\vec{a}$ and $\vec{b}\times\vec{c}$

vi) A vector in the plane determined by $\vec{a}$ and $\vec{b}$, and perpendicular (垂直) to $\vec{c}$

8. Suppose that $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ are vectors in $\mathbb{R}^3$. Indicate which of the following expressions are vectors, which are scalars, and which are nonsense

i) $(\vec{a} \times \vec{b}) \times \vec{c}$

ii) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

iii) $(\vec{a} \cdot \vec{b}) \times (\vec{c} \cdot \vec{d})$

iv) $(\vec{a} \times \vec{b}) \cdot \vec{c}$

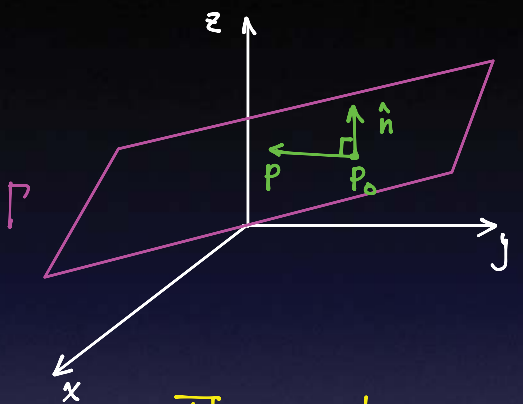
v) $(\vec{a} \cdot \vec{b}) \times (\vec{c} \times \vec{d})$

vi) $\vec{a} \times [(\vec{b} \cdot \vec{c})\, \vec{d}\,]$

vii) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

viii) $(\vec{a} \cdot \vec{b})\, \vec{c} - (\vec{a} \times \vec{b})$

§ 1.5 Equations for planes; Distance problems



A plane in $\mathbb{R}^3$:

i) $P_0(x_0, y_0, z_0) \in$ the plane Γ

ii) $\vec{n} = A\hat{i} + B\hat{j} + c\hat{k}$ is perpendicular (normal) to the plane

The vector equation of the plane:

$$\boxed{\vec{n} \cdot \overrightarrow{P_0P} = 0}$$

$$\Rightarrow (A\hat{i} + B\hat{j} + c\hat{k}) \cdot ((x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}) = 0$$

or $\boxed{A(x - x_0) + B(y - y_0) + c(z - z_0) = 0}$

$\sim Ax + By + cz = D$, $(D = Ax_0 + By_0 + Cz_0)$

Ex. The plane through the point $(3,2,1)$ with normal vector $2\hat{i}-\hat{j}+4\hat{k}$:

$$(2\hat{i}-\hat{j}+4\hat{k})\cdot((x-3)\hat{i}+(y-2)\hat{j}+(z-1)\hat{k})=0$$

$$\Rightarrow 2(x-3)-(y-2)+4(z-1)=0$$

or $2x-y+4z=8$

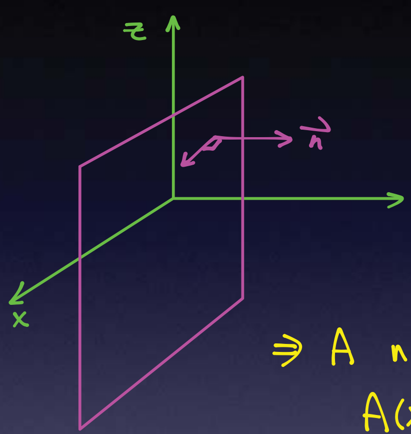
Ex: Given the plane with equation $7x+2y-3z=1$, find a normal vector to the plane and identify three point that lie on the plane.

$\vec{n}=7\hat{i}+2\hat{j}-3\hat{k}$ (or any vector parallel to it.)

Take $y=z=0 \Rightarrow 7x=1$, $x=1/7$ ~ $(\frac{1}{7},0,0)$

Similarly, $(0,\frac{1}{2},0)$ and $(0,0,-1/3)$ lie on the plane.

Ex. Let the z-axis points vertically in $\mathbb{R}^3$, what is the general form of the equation of a non-vertical plane?



Vertical plane ~ a horizontal $\vec{n}$ (normal vector) : $\vec{n}=(A,B,0)$

$$\Rightarrow A(x-x_0)+B(y-y_0)+0(z-z_0)=0$$

$\Rightarrow$ A non-vertical plane:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0, \quad \boxed{C\neq 0}$$

Ex. A plane is determined by 3 (noncollinear) points.

$P_0(1,2,0)$, $P_1(3,1,2)$, $P_2(0,1,1)$

The plane that contains the above points:

Method 1: The equation of the plane: $Ax+By+Cz=D$

P_0: $A+2B=D$

P_1: $3A+B+2C=D$ ← 3 equations and 4 unknowns

P_2: $B+C=D$

either no solution or infinitely many solutions

⇑ our case!

(A, B, C, D) is a solution

$\Rightarrow \alpha$ (A, B, C, D) is also a sol.

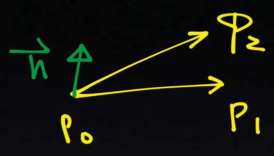
$\Rightarrow A=\frac{-1}{7}D, \quad B=\frac{4}{7}D, \quad C=\frac{3}{7}D$

$\Rightarrow (-x+4y+3z)\frac{1}{7}D = D$

$D\neq 0$ (Otherwise, we get $A=B=C=0$!)

$\Rightarrow x-4y-3z=-7$

Method 2: Use vectors



$\overrightarrow{P_0P_1} = 2\hat{i}-\hat{j}+2\hat{k}$

$\overrightarrow{P_0P_2} = -\hat{i}-\hat{j}+\hat{k}$

$$\vec{n} = \overrightarrow{P_0P_1}\times\overrightarrow{P_0P_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \hat{i}-4\hat{j}-3\hat{k}$$

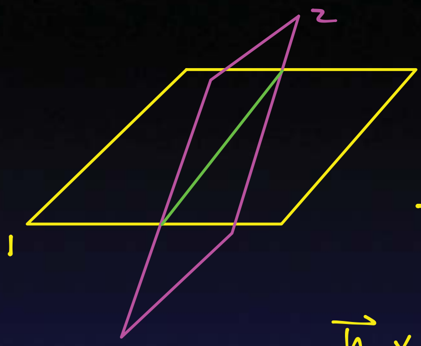
$\Rightarrow x-4y-3z=D$

P_0: (1, 2, 0) 代入: $1-8+0=D=-7 \Rightarrow x-4y-3z=-7$ #

Ex. Consider the two planes having equations $x-2y+z=4$ and $2x+y+3z=-7$. This equation of the line of their intersection:

1. $\begin{cases} x-2y+z=4 \\ 2x+y+3z=-7 \end{cases} \Rightarrow \begin{array}{l} x=x(z) \\ y=y(z) \end{array}$ or $\begin{cases} x=x(t) \\ y=y(t) \\ z=t \end{cases}$

2. Use vector cross product:



$\vec{n}_1 = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{n}_2 = 2\hat{i} + \hat{j} + \hat{k}$

The vector parallel to the line:

$$\vec{n}_1 \times \vec{n}_2 = (-2\cdot 3 - 1\cdot 1)\,i + (1\cdot 2 - 1\cdot 3)\,\hat{j} + (1\cdot 1 - (-2)\cdot 2)\,\hat{k}$$

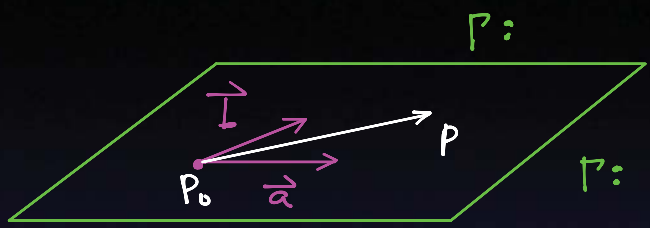
$$= -7\hat{i} - \hat{j} + 5\hat{k}$$

$$\Rightarrow \vec{r}(t) = P_0 + t(-7\hat{i} - \hat{j} + 5\hat{k})$$

P_0: Take $z = 0$, $\begin{cases} x - 2y = 4 \\ 2x + y = -7 \end{cases} \Rightarrow x = -2,\ y = -3 \Rightarrow (-2, -3, 0)$

$$\Rightarrow \begin{cases} x = -7t - 2 \\ y = -t - 3 \\ z = 5t \end{cases}$$

△ Parametric equations of planes:



$\vec{x} = \overrightarrow{P_0P}$

$P: \{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} = s\vec{a} + t\vec{b},\ s, t \in \mathbb{R} \}$

$\vec{op} \equiv \vec{r} \Rightarrow \boxed{\vec{r}(s,t) = \vec{oP_0} + s\vec{a} + t\vec{b}}$ ← two parameters

In terms of x, y, z: $\begin{cases} x = sa_1 + tb_1 + x_0 \\ y = sa_2 + tb_2 + y_0 \\ z = sa_3 + tb_3 + z_0 \end{cases}$ ← $P_0: (x_0, y_0, z_0)$

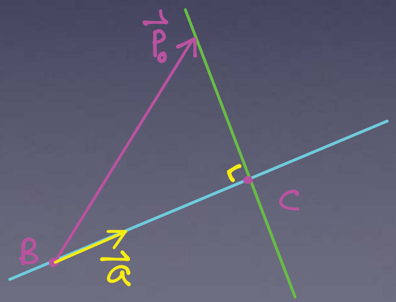
Ex: The plane that passes through the point $(1, 0, -1)$ and is parallel to the vectors $3\hat{i} - \hat{k}$ and $2\hat{i} + 5\hat{j} + 2\hat{k}$:

$$\vec{r} = s(3\hat{i} - \hat{k}) + t(2\hat{i} + 5\hat{j} + 2\hat{k}) + (\hat{i} + 0\hat{j} - \hat{k})$$

$$\Rightarrow \begin{cases} x = 3s + 2t + 1 \\ y = 5t \\ z = 2t - s - 1 \end{cases}$$

△ Distance problems: (投影觀念之應用)

Ex. The distance between a point and a line:



distance $= |\overrightarrow{CP_0}|$

$\overrightarrow{CP_0} = \overrightarrow{BP_0} - \overrightarrow{BC}$, $\overrightarrow{BC} = \text{Proj}_{\vec{a}}\overrightarrow{BP_0}$

The point P_0: $(2,1,3)$

The line: $\vec{r}(t) = t\underbrace{(-1,1,2)}_{\vec{a}} + \underbrace{(2,3,-2)}_{\text{taken as } B \,:\, \overrightarrow{OB} = \vec{r}(t=0)}$

$\overrightarrow{BP_0} = (0,-2,5)$

$$\overrightarrow{BC} = \text{Proj}_{\vec{a}}\overrightarrow{BP_0} = \left(\overrightarrow{BP_0} \cdot \hat{a}\right)\hat{a} = \frac{\overrightarrow{BP_0} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}$$

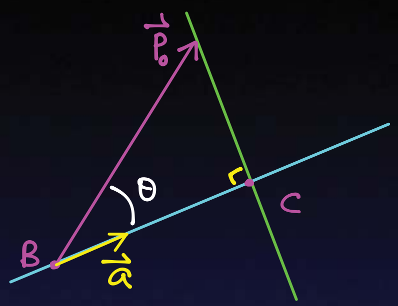
$$= \frac{(0,-2,5)\cdot(-1,1,-2)}{(-1,1,-2)\cdot(-1,1,-2)}(-1,1,-2) = \frac{-12}{6}(-1,1,-2)$$

$$= (+2,-2,+4)$$

$$\Rightarrow \overrightarrow{CP_0} = \overrightarrow{BP_0} - \overrightarrow{BC} = (0,-2,5) - (+2,-2,+4) = (-2,0,1)$$

$$\Rightarrow |\overrightarrow{CP_0}| = \sqrt{(-2)^2 + 0^2 + 1^2} = \sqrt{5} \ \#$$

You may also calculate it as follows: (specific to 3D)



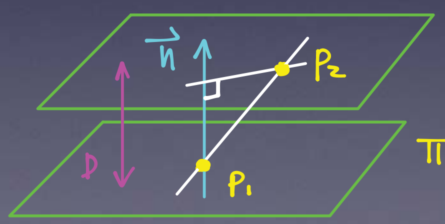
$$|\overrightarrow{CP_0}| = |\overrightarrow{BP_0}| \sin\theta$$

$$|\overrightarrow{BP_0} \times \vec{a}| = |\overrightarrow{BP_0}|\,|\vec{a}| \sin\theta$$

From $\overrightarrow{BP_0} \times \vec{a} = \hat{i} + 5\hat{j} + 2\hat{k}$

$$\Rightarrow |\overrightarrow{BP_0}| \sin\theta = \frac{|\overrightarrow{BP_0} \times \vec{a}|}{|\vec{a}|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

Ex: Distance between parallel planes:



$\Pi_2: 2x - 2y + z = 20 \quad \Rightarrow \vec{n} = 2\hat{i} - 2\hat{j} + \hat{z}$

$\Pi_1: 2x - 2y + z = 5 \quad$ Take $x = y = 0$,

$\Rightarrow (0, 0, 5) \in \Pi_1 \sim P_1$

$(0, 0, 20) \in \Pi_2 \sim P_2$

$$\overrightarrow{P_1P_2} = (0, 0, 15)$$

$$\mathrm{Proj}_{\vec{n}} \overrightarrow{P_1P_2} = (\overrightarrow{P_1P_2} \cdot \hat{n})\,\hat{n} = \left(\frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}\right)\vec{n} = \frac{(0,0,15)\cdot(2,-2,1)}{(2,-2,1)\cdot(2,-2,1)}(2,-2,1)$$

$$= \frac{-15}{9}(2,-2,1) = \frac{-5}{3}(2,-2,1)$$

$$D = |\mathrm{Proj}_{\vec{n}} \overrightarrow{P_1P_2}| = \frac{5}{3} \cdot \sqrt{9} = 5 \quad \#$$

Ex: Distance between two <u>skew</u> lines (非平行且不相交)

$\vec{l}_1(t) = t(2,1,3) + (0,5,-1)$, $\vec{l}_2(t) = t(1,-1,0) + (-1,2,0)$

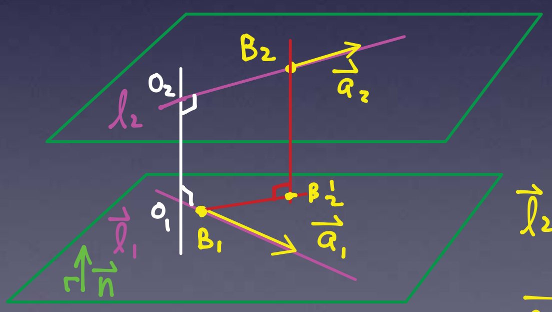
$\vec{l}_1(t_1) = \vec{l}_2(t_2)$: No solution $\sim$ No intersection

$$\begin{cases} 2t_1 = t_2 - 1 & \longrightarrow t_2 = 2t_1 + 1 = \frac{5}{3} \\ t_1 + 5 = -t_2 + 2 & \\ 3t_1 - 1 = 0 & \longrightarrow t_1 = 1/3 \end{cases}$$

兩線之間的距離 = 2線上的點最近的距離

$= |\vec{l}_1(t_1) - \vec{l}_2(t_2)|$ 的極小值 $\longrightarrow$ calculus

幾何觀點：找一組分別包含 $\vec{l}_1$ 與 $\vec{l}_2$，且彼此相互平行的平面，則此二平面間的距離，即為兩線間的距離



$\vec{l}_1 = t\vec{a}_1 + \vec{B}_1$

normal vector $\vec{n} = \vec{a}_1 \times \vec{a}_2$

$\vec{l}_2 = t\vec{a}_2 + \vec{B}_2$

$\overline{O_1O_2} = \overline{B_2'B_1}$

$\overrightarrow{B_2'B_2} = \text{Proj}_{\vec{n}}\, \overrightarrow{B_1B_2}$

$\overrightarrow{B_1B_2} = (-1,2,0) - (0,5,-1) = (-1,-3,1)$

$\vec{n} = \vec{a}_1 \times \vec{a}_2 = (2,1,3) \times (1,-1,0) = (1\cdot 0 - 3\cdot(-1),\ 3\cdot 1 - 2\cdot 0,\ 2\cdot(-1) - 1\cdot 1)$

$= (+3, 3, -3) = 3\,(1,1,-1) \equiv 3\,\vec{n}'$

$$\text{Proj}_{\vec{n}}\, \overrightarrow{B_1B_2} = \frac{\vec{n}' \cdot \overrightarrow{B_1B_2}}{\vec{n}' \cdot \vec{n}'}\,\vec{n}' = \frac{(1,1,-1)\cdot(-1,-3,1)}{(1,1,-1)\cdot(1,1,-1)}\,(1,1,-1)$$

$$= \frac{-5}{3}\,(1,1,-1)$$

$$\text{Distance} = \frac{5}{3}\sqrt{1^2+1^2+(-1)^2} = \frac{5}{3}\sqrt{3}$$

#

□ Exercises :

1. Find the equation for the plane that

a) containing the point $(3,-1,2)$ and perpendicular to $\hat{i}-\hat{j}+2\hat{k}$

b) containing the point $(3,-1,2)$, $(2,0,5)$ and $(1,-2,4)$

c) passes through the point $(2,-1,2)$ and is parallel to the plane $5x-4y+z=1$

d) contains two lines, l_1: $x=t+2$, $y=3t-5$, $z=5t+1$

l_2: $x=5-t$, $y=3t-10$, $z=9-2t$

(Make sure that these two lines intersect first !)

2. The plane, given parametrically by $x=3s-t+2$, $y=4s+t$, $z=s+5t+3$, satisfies the equation $Ax+By+Cz=D$. Find A, B, C, D

3. Find the distance between

a) the two planes : $x-3y+2z=1$, $x-3y+2z=8$

b) the point $(-11,10,20)$ and the line : $x=5-t$, $y=3$, $z=7t+8$

c) the two lines :

$\vec{l}_1(t)=(t-7)\hat{i}+(5t+1)\hat{j}+(3-2t)\hat{k}$, $\vec{l}_2(t)=4t\hat{i}+(2-t)\hat{j}+(8t+1)\hat{k}$

4. Show that the distance d between the two parallel planes $Ax+By+Cz=D_1$ and $Ax+By+Cz=D_2$ is

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2+B^2+C^2}}$$

5. Suppose that two lines: $\vec{l}_1(t) = \vec{a}_1 t + \vec{b}_1$, $\vec{l}_2(t) = \vec{a}_2 t + \vec{b}_2$ are skew lines in $\mathbb{R}^3$. Show that the distance between the two lines is given by

$$D = \frac{|(\vec{a}_1 \times \vec{a}_2)\cdot(\vec{b}_2 - \vec{b}_1)|}{|\vec{a}_1 \times \vec{a}_2|}$$

§1.6 Some n-dimensional Geometry, matrices and determinants

Definition: A vector in $\mathbb{R}^n$ is an ordered n-tuple of real numbers.

No arrows, No $\sin\theta$, $\cos\theta$ ... for $n>3$

$\mathbb{A} = (a_1, a_2, \cdots a_n)$ ~ a vector in $\mathbb{R}^n$

$\mathbb{B} = (b_1, b_2, \cdots b_n)$ ~ another vector in $\mathbb{R}^n$

$\mathbb{A} = \mathbb{B} \iff a_i = b_i$, $i = 1, 2, \cdots n$

$\mathbb{A} + \mathbb{B} = (a_1+b_1, a_2+b_2, \cdots a_n+b_n)$

$k\,\mathbb{A} = (ka_1, ka_2, \cdots ka_n)$. $k \in \mathbb{R}$

$\mathbb{A}\cdot\mathbb{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \in \mathbb{R}$

$|\mathbb{A}| \equiv \sqrt{\mathbb{A}\cdot\mathbb{A}}$, With the above definitions,

- The distance between two points in $\mathbb{R}^n = |\vec{r}_1 - \vec{r}_2|$
- $\mathbb{A}\cdot\mathbb{B} = 0 \iff \mathbb{A}$ is orthogonal to $\mathbb{B}$

  (We may still define $\theta = \cos^{-1}\dfrac{\mathbb{A}\cdot\mathbb{B}}{|\mathbb{A}||\mathbb{B}|}$ as the angle between $\mathbb{A}$ and $\mathbb{B}$ since $-1 \leq \dfrac{\mathbb{A}\cdot\mathbb{B}}{|\mathbb{A}||\mathbb{B}|} \leq 1$ $\leftarrow$ will be proved later !)

- Standard basis vectors in $\mathbb{R}^n$:

$$\hat{e}_1 = (1,0,0,\cdots 0),\quad \hat{e}_2 = (0,1,0,\cdots 0),\quad \cdots\quad \hat{e}_n = (0,0,0,\cdots 1)$$

$$\mathbb{A} = (a_1, a_2, a_3 \cdots) = a_1\hat{e}_1 + a_2\hat{e}_2 + \cdots + a_n\hat{e}_n$$

$$= \sum_{i=1}^{n} a_i\hat{e}_i$$

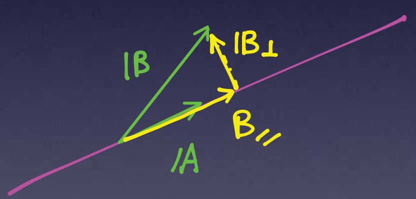
- Note that there is no simple generalization of the cross product for $n \neq 3$.

△ Two famous inequalities :

1. The Cauchy – Schwarz inequality :

   For all vectors $\mathbb{A}$ and $\mathbb{B}$ in $\mathbb{R}^n$, we have

$$\boxed{|\mathbb{A}\cdot\mathbb{B}| \leq |\mathbb{A}|\,|\mathbb{B}|}$$



Geometric picture (even though $n > 3$)

$$\mathbb{B} = \mathbb{B}_{//} + \mathbb{B}_{\perp} = \mathrm{Proj}_{\mathbb{A}}\mathbb{B} + \mathbb{B}_{\perp}$$

$$= \frac{\mathbb{B}\cdot\mathbb{A}}{|\mathbb{A}|\,|\mathbb{A}|}\mathbb{A} + \mathbb{B}_{\perp}$$

Note that : $\mathbb{B}_{\perp} \equiv \mathbb{B} - \mathbb{B}_{//} \Rightarrow \mathbb{B}_{\perp}\cdot\mathbb{A} = \mathbb{B}\cdot\mathbb{A} - \dfrac{\mathbb{B}\cdot\mathbb{A}}{|\mathbb{A}|\,|\mathbb{A}|}\underbrace{\mathbb{A}\cdot\mathbb{A}}_{|\mathbb{A}|^2} = 0$

This confirms that $\mathrm{Proj}_{\mathbb{A}}\mathbb{B}$ is indeed the projection of $\mathbb{B}$ on $\mathbb{A}$.

Now, $|\mathbb{B}_\perp|^2 \geq 0$ (We use $B = |\mathbb{B}|$, $A = |\mathbb{A}|$)

$$\Rightarrow \left(\mathbb{B} - \frac{\mathbb{B}\cdot\mathbb{A}}{A^2}\mathbb{A}\right)\cdot\left(\mathbb{B} - \frac{\mathbb{B}\cdot\mathbb{A}}{A^2}\mathbb{A}\right)$$

$$= \underbrace{\mathbb{B}\cdot\mathbb{B}}_{B^2} - \frac{(\mathbb{B}\cdot\mathbb{A})^2}{A^2}\cdot 2 + \frac{\mathbb{B}\cdot\mathbb{A}}{A^4}\underbrace{\mathbb{A}\cdot\mathbb{A}}_{A^2} \geq 0$$

$$\Rightarrow \quad B^2 \geq \frac{(\mathbb{B}\cdot\mathbb{A})^2}{A^2}$$

or $\quad |\mathbb{B}|\,|\mathbb{A}| \geq |\mathbb{B}\cdot\mathbb{A}|$ #

When does the equality sign hold true?

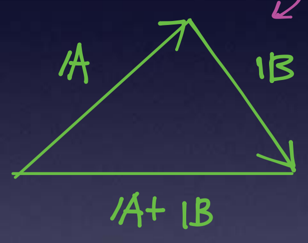
$|\mathbb{B}_\perp|^2 = 0 \Rightarrow \mathbb{B}_\perp$ is a null vector, i.e. $\mathbb{B} \parallel \mathbb{A}$ or $\mathbb{B} = c\,\mathbb{A}$

2. The triangle inequality:

For any vectors $\mathbb{A}$ and $\mathbb{B}$ in $\mathbb{R}^n$, we have

$$\boxed{|\mathbb{A}+\mathbb{B}| \leq |\mathbb{A}| + |\mathbb{B}|}$$

Geometric picture:



$$(A+B)^2 = (\mathbb{A}+\mathbb{B})\cdot(\mathbb{A}+\mathbb{B})$$
$$= \mathbb{A}\cdot\mathbb{A} + \mathbb{B}\cdot\mathbb{B} + 2\,\mathbb{A}\cdot\mathbb{B}$$
$$= A^2 + B^2 + 2\,\mathbb{A}\cdot\mathbb{B}$$
$$\leq A^2 + B^2 + 2AB \quad \leftarrow \text{Schwarz inequality}$$

$$\Rightarrow |\mathbb{A}+\mathbb{B}|^2 \leq \left(|\mathbb{A}| + |\mathbb{B}|\right)^2$$

or $\quad |\mathbb{A}| + |\mathbb{B}| \geq |\mathbb{A}+\mathbb{B}|$ #

"$=$" $\sim$ $\mathbb{A} = k\,\mathbb{B}$, $k \in \mathbb{R}$

△ Matrices

A $m \times n$ matrix : m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (\text{shorthand})\ (a_{ij})$$

Vectors in $\mathbb{R}^n$ ~ a $1 \times n$ matrix or a row matrix / vector

$$\mathbf{a} = (a_1 \quad a_2 \quad \cdots \quad a_n)$$

or more typically, a column matrix / vector

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \text{a } n \times 1 \text{ matrix}$$

– A $m \times n$ matrix = a "vector of vectors"

~ m row vectors in $\mathbb{R}^n$ $\quad [a_{11} \cdots a_{1n}]$, $[a_{21} \cdots a_{2n}]$, $\vdots$

~ n column vectors in $\mathbb{R}^m$ $\quad \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots$

△ Matrix addition : A and B are two $m \times n$ matrices,

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 0 & -1 \\ -2 & 5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 1 \\ 5 & 3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 8 & 2 & 2 \\ 2 & 10 & 6 \end{bmatrix}, \quad A+C \text{ is not defined!}$$

– $A+B = B+A$

– $(A+B)+C = A+(B+C)$

△ Scalar multiplication : $k \in \mathbb{R}$, $(kA)_{ij} = k A_{ij}$

$\Rightarrow$ – $(k+l)A = kA + lA$

– $k(A+B) = kA + kB$

– $k(lA) = (kl)A = l(kA)$

$k, l \in \mathbb{R}$

A, B are $m \times n$ matrices

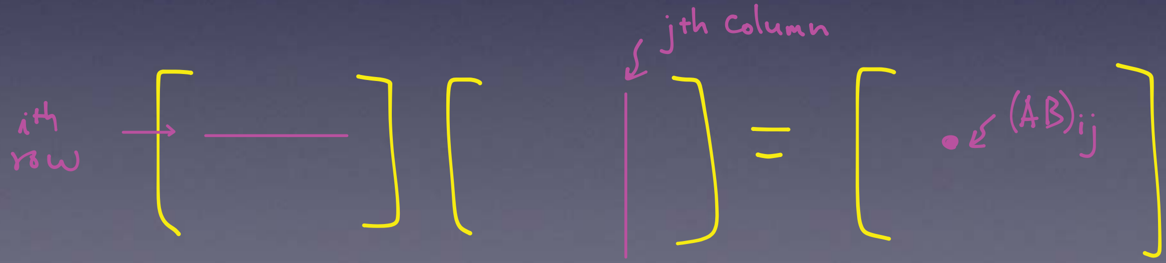
△ Matrix multiplication

A is a $m \times n$ matrix, B is a $n \times p$ matrix. The matrix product AB is a $m \times p$ matrix such that

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \qquad \begin{array}{l} i = 1, 2, \cdots m \\ j = 1, 2, \cdots p \end{array}$$

$$= \sum_{k=1}^{n} A_{ik} B_{kj}$$

= the inner between the ith row vector in A and the jth column vector in B



Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 7 & 0 \\ 2 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 7 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 13 \\ 47 & 28 \end{bmatrix}$$

$(AB)_{12}$: $1 \cdot 1 + 2 \cdot 0 + 3 \cdot 4 = 13$

$(BA)_{23}$

$$BA = \begin{bmatrix} 0 & 1 \\ 7 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 14 & 21 \\ 18 & 24 & 30 \end{bmatrix}$$

$7 \cdot 3 + 0 \cdot 6 = 21$

Note that for matrix product, we have in general

$$\boxed{AB \neq BA}$$, and

- $(AB)C = A(BC)$
- $k(AB) = (kA)B = A(kB)$, $k \in \mathbb{R}$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$

△ The transpose of a matrix (轉置矩陣)

The transpose of a $m\times n$ matrix A is a $n\times m$ matrix such that

$$\boxed{(A^T)_{ij} = A_{ji}}$$

$(A^T)_{12}$ $\quad (A^T)_{11} = A_{11},\ (A^T)_{12} = A_{21}$

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\quad (A^T)_{32} = A_{23} \ldots$ etc.

A_{21}

— $(A^T)^T = A \quad \left((A^T)^T_{ij} = A^T_{ji} = A_{ij} \right)$

— $\boxed{(AB)^T = B^T A^T}$ $\quad \left((AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} \right.$ with $B_{ki} = (B^T)_{ik}$, $A_{jk} = (A^T)_{kj}$

$$\left. = \sum_k (B^T)_{ik} (A^T)_{kj} = (B^T \cdot A^T)_{ij} \right)$$

Note that the inner product between two vectors can be written in matrix form:

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = [a_1\ a_2 \cdots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a^T b$$

△ A linear mapping in $\mathbb{R}^n$:

A linear function of a single variable: $f(x) = ax$

Generalization to $\mathbb{R}^n$: $\vec{x} \in \mathbb{R}^n$

$F: \mathbb{R}^n \to \mathbb{R}^m$, $F(\vec{x}) = A x \quad A$: a constant $m\times n$ matrix

$$= \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}$$

linear: $F(k\vec{x}) = k F(\vec{x}) \ \sim\ A(kx) = k(Ax)$

$F(\vec{x}+\vec{y}) = F(\vec{x}) + F(\vec{y}) \quad k(x+Y) = kX + kY$

matrix multiplication = the composition of linear mappings:

$G: \mathbb{R}^m \to \mathbb{R}^p \quad G(y) = \underset{p\times m}{B}\,\underbrace{y}_{m\times 1}\,, \qquad F: \mathbb{R}^n \to \mathbb{R}^m \quad F(x) = \underbrace{A}_{m\times n}\,\underbrace{X}_{n\times 1}$

$$\Rightarrow G\circ F(x) = G(F(x)) = G(AX) = B(AX) = (BA)x$$

i.e. $G\circ F$ is a linear mapping from $\mathbb{R}^n \to \mathbb{R}^p$

$\triangle$ A hyperplane in $\mathbb{R}^n$ $(n>3)$

$A_1(x_1-b_1) + A_2(x_2-b_2) + \cdots + A_n(x_n-b_n) = 0$ vs. $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$\sim$ a $(n-1)$ dimensional object

$\sim$ $\mathbb{A}\cdot(\mathbb{X}-\mathbb{b}) = 0$ $\quad \mathbb{A} = (A_1, A_2 \cdots A_n)$, $\mathbb{X} = (x_1, x_2, \cdots x_n)$, $\mathbb{b} = (b_1, b_2, \cdots b_n)$

"normal vector"

$\triangle$ The determinant of a $n\times n$ matrix:

2×2 : $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

3×3 : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ C_{11} $\qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $-C_{12}$ $\qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $+C_{13}$

$\Rightarrow$ cofactor

Cofactor C_{ij} of a_{ij} :

The cofactor $C_{ij} \equiv (-1)^{i+j} M_{ij}$

M_{ij} = the determinant of the submatrix obtained by eliminating the ith row and jth column of A.

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = \sum_{i=1}^{3} a_{1i} C_{1i}$$

We may generalize this definition to any $n \times n$ matrix :

Ex $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ -2 & 1 & 0 & 5 \\ 4 & 2 & -1 & 0 \\ 3 & -2 & 1 & 1 \end{bmatrix}$ is a 4×4 matrix

$$\det A \equiv \sum_{i=1}^{4} a_{1i} C_{1i} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 0 & 5 \\ 2 & -1 & 0 \\ -2 & 1 & 1 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} -2 & 0 & 5 \\ 4 & -1 & 0 \\ 3 & 1 & 1 \end{vmatrix}$$

$$+ 1 \cdot (-1)^{1+3} \begin{vmatrix} -2 & 1 & 5 \\ 4 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} + 3 \cdot (-1)^{1+4} \begin{vmatrix} -2 & 1 & 0 \\ 4 & 2 & -1 \\ 3 & -2 & 1 \end{vmatrix} = -132$$

Δ Properties of determinants

1. Transpose : $\boxed{\det(A^T) = \det A}$

Ex : $A = \begin{pmatrix} 1 & 28 & -29 \\ 0 & 1 & -4 \\ 0 & -2 & 5 \end{pmatrix}$, $\det A = \det A^T = \begin{vmatrix} 1 & 0 & 0 \\ 28 & 1 & -2 \\ -29 & -4 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = -3$

2. Scale factor : If every element of any row or column of the matrix A is multiplied by a scalar k, then the determinant is $k \det A$

Ex. $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ k a_{11} & k a_{12} & k a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta = \begin{vmatrix} -1 & 99 & 1 \\ 2 & 33 & -2 \\ 3 & 55 & 1 \end{vmatrix} = 11 \cdot \begin{vmatrix} -1 & 9 & 1 \\ 2 & 3 & -2 \\ 3 & 5 & 1 \end{vmatrix} = -924$$

3. Row/Column exchange:

If B is obtained from A by interchanging two rows (or columns), then $\det B = -\det A$

$\Rightarrow$ If A is a $n \times n$ matrix with two identical rows/columns

$\det A = 0$

Ex $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$

Ex:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ -1 & 3 & 0 & 4 \\ -1 & 2 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 0 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ -1 & 3 & 0 & 4 \\ -1 & 2 & 0 & -1 \end{vmatrix}$$

$$= (-1) \cdot (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 1 & 1 & 2 \\ -1 & 0 & 4 \\ -1 & 0 & -1 \end{vmatrix} = 10$$

4. Expansion by any row or column (from 1. and 3.)

Ex: A is a 3×3 matrix

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$
$$= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33} \leftarrow \text{3th row}$$
$$= a_{12} C_{12} + a_{22} C_{22} + a_{32} C_{32} \leftarrow \text{2nd column}$$

5. If the matrix B is constructed from A by adding k times one row (or column) to another row (column) then $\det B = \det A$

Ex: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\times k$, $+$

$$\det B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + k a_{11} & a_{32} + k a_{12} & a_{33} + k a_{13} \end{vmatrix}$$

$$\det B = (a_{31} + k a_{11}) C_{31} + (a_{32} + k a_{12}) C_{32} + (a_{33} + k a_{13}) C_{33}$$

$$= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33} \Rightarrow \det A$$

$$+ k \underbrace{(a_{11} C_{31} + a_{12} C_{32} + a_{13} C_{33})}_{-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = 0 \ !}$$

$$C_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = -\begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix}, \quad C_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = +\begin{vmatrix} a_{21} & a_{23} \\ a_{11} & a_{13} \end{vmatrix}$$

$$C_{33} = +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}$$

$\Rightarrow \det B = \det A$

5.1 If the rows or columns of A are linearly dependent, then $\det A = 0$

6. A and B are two $n \times n$ matrices,

$$\boxed{\det(AB) = \det A \cdot \det B}$$

7. The inverse of a $n \times n$ matrix:

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{we define } \operatorname{adj} A \equiv \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

That is, $(\operatorname{adj} A)_{ij} = C_{ji}$

$$(A \cdot \operatorname{adj} A)_{ij} = \sum_{k=1}^{n} a_{ik} (\operatorname{adj} A)_{kj} = \sum_{k=1}^{n} a_{ik} \cdot C_{jk}$$

For $i = j$ (say, $= 2$)

$$\sum_{k=1}^{3} a_{2k} C_{2k} = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$= \det A$ (Expanded by the 2nd row)

For $i \neq j$ (say, $i=1$, $j=2$)

$$\sum_{k=1}^{3} a_{1k} C_{2k} = a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \longleftarrow \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

replace the 2nd row by $[a_{11} \quad a_{12} \quad a_{13}]$

$$= 0$$

$$\Rightarrow A \cdot \operatorname{adj} A = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = (\det A) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (\det A) \cdot I$$ ← Identity or unit matrix

單位矩陣

Definition: The inverse of A, denoted by A^{-1}, is a matrix such that

$$A \cdot A^{-1} = I = A^{-1} \cdot A$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{\det A} (\operatorname{adj} A)} \iff \det A = 0 \sim A^{-1} \text{ does not exist!}$$

Ex: $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

$\Rightarrow \det A = -4$ (A^{-1} exists!)

$C_{11} = -3$, $C_{12} = -1$, $C_{13} = -1$

$C_{21} = 5$, $C_{22} = -1$, $C_{23} = 3$

$C_{31} = -1$, $C_{32} = 1$, $C_{33} = 1$

$$\Rightarrow A^{-1} = \frac{-1}{4} \begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Exercise:

1. Let $A = \begin{bmatrix} 1 & 2 & k \\ k & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. Find $\det A$, and the value of k for which $\det A = 0$

2. The determinant $D_n = \begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & x \end{vmatrix}$ has n rows.

   Factorize D_n in terms of x

3. Find all values of x for which $\begin{vmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix} = 0$

4. Use property 6 to show that

   $\det(A^{-1}) = \dfrac{1}{\det A}$, $\det(\operatorname{adj} A) = (\det A)^{n-1}$

   $[\ \det[kA] = k^n(\det A)\ ]$