Quantum Physics（量子物理）習題

Robert Eisberg（Second edition）

CH 06：Solutions of time-independent Schroedinger equations

6-01、Show that the step potential eigenfunction, for $E < V_0$, can be converted in form from the sum of two traveling waves, as in (6-24), to a standing wave, as in (6-29).

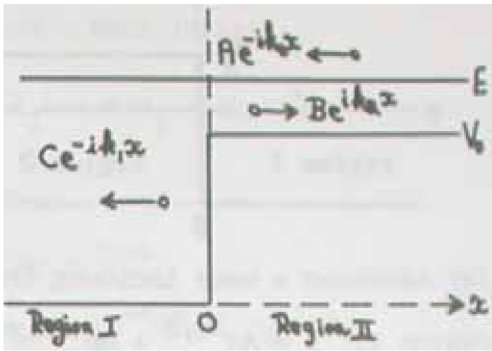
<解>：

6-02、Repeat the step potential calculate of Section 6-4, but with the particle initially in the region $x > 0$ where $V(x) = V_0$, and traveling in the direction of decreasing x towards the point $x = 0$ where the potential steps down to its value $V(x) = 0$ in the region $x < 0$. Show that the transmission and reflection coefficients are the same as those obtained in Section 6-4.

<解>：Assume that

$$\psi_1 = Ce^{-ik_1x}$$

$$\psi_2 = Ae^{-ik_2x} + Be^{ik_2x}$$



Where A=amplitude of incident wave

B=amplitude of reflected wave

C=amplitude of transmitted wave

There is no wave moving in the +x-direction in region I.

Also, $k_1 = \dfrac{(2mE)^{1/2}}{\hbar}$, $k_2 = \dfrac{\{2m(E-V_0)\}^{1/2}}{\hbar}$

Continuity of wavefunction and derivative at $x = 0$ imply $A + B = C$, $-k_2A + k_2B = -k_1C$

These equations may be solved to give the reflection and the transmission amplitudes in terms of the incident amplitude, the results being: $B = \dfrac{k_2 - k_1}{k_2 + k_1}A$;

$$C = \frac{2k_2}{k_2 + k_1}A$$

The reflection coefficient R and transmission coefficient T now become

$$R=\frac{B^*B}{A^*A}=\frac{B^2}{A^2}=(\frac{k_1-k_2}{k_1+k_2})^2$$

$$T=\frac{v_1C^*C}{v_2A^*A}=(\frac{\hbar k_1}{\hbar k_2})(\frac{2k_2}{k_1+k_2})^2=\frac{4k_1k_2}{(k_1+k_2)^2}$$

These expressions for R and T are the same as those obtained if the incident wave came from the left…….##

6-03、Prove (6-43) stating that the sum of the reflection and transmission coefficients equals one, for the case of a step potential with $E>V_0$.

<解>：

6-04、Prove (6-44) which expresses the reflection and transmission coefficients in terms of the ratio $\frac{E}{V_0}$.

<解>：

6-05、Consider a particle tunneling through a rectangular potential barrier. Write the general solutions presented in Section 6-5, which give the form of ψ in the different regions of the potential. (a) Then find four relations between the five arbitrary constants by matching ψ and $d\psi/dx$ at the boundaries between these regions. (b) Use these relations to evaluate the transmission coefficient T, thereby verifying (6-49). (Hint：First eliminate F and G, leaving relations between A, B, and C. Then eliminate B.)

<解>：

6-06、Show that the expression of (6-49), for the transmission coefficient in tunneling through a rectangular potential barrier, reduces to the form quoted in (6-50) if the exponents are very large.

<解>：If $k_2a>>1$, then $e^{k_2a}>>e^{-k_2a}$ and the transmission coefficient becomes, under

these circumstances, $T = \{1 + \frac{e^{2k_2 a}}{16\frac{E}{V_0}(1-\frac{E}{V_0})}\}^{-1}$.

Now $0 < \frac{E}{V_0} < 1$ and therefore $16\frac{E}{V_0}(1-\frac{E}{V_0}) \le 4$, the upper limit occurring at $\frac{E}{V_0} = \frac{1}{2}$.

Hence, if $e^{2k_2 a} > 4$, $\frac{e^{2k_2 a}}{16\frac{E}{V_0}(1-\frac{E}{V_0})} > 1$.

Since, in fact, it is assumed that $e^{2k_2 a} >> 1$, $\frac{e^{2k_2 a}}{16\frac{E}{V_0}(1-\frac{E}{V_0})} >> 1$,

And therefore, under these conditions, $T = 16\frac{E}{V_0}(1-\frac{E}{V_0})e^{-2k_2 a}$ ....##

6-07、Consider a particle passing over a rectangular potential barrier. Write the general solutions, presented in Section 6-5, which give the form of ψ in the different regions of the potential. (a) Then find four relations between the five arbitrary constants by matching ψ and $\frac{d\psi}{dx}$ at the boundaries between these regions. (b) Use these relations to evaluate the transmission coefficient T, thereby verifying (6-51). (Hint : Note that the four relations become exactly the same as those found in the fires part of Problem 5, if k_{II} is replaced by ik_{III}. Make this substitution in (6-49) to obtain directly (6-51).)

<解>：

6-08、(a) Evaluate the transmission coefficient for an electron of total energy $2eV$ incident upon a rectangular potential barrier of height $4eV$ and thickness $10^{-10}m$, using (6-49) and then using (6-50). Repect the evaluation for a barrier thickness of (b) $9\times10^{-9}m$ and (c) $10^{-9}m$.

<解>：(a) 0.62

(b) 1.07×10^{-56}

(c) 2.1×10^{-6}

6-09、A proton and a deuteron (a particle with the same charge as a proton, but twice the mass) attempt to penetrate a rectangular potential barrier of height $10MeV$ and thickness $10^{-14}m$. Both particle have total energies of $3MeV$. (a) Use qualitative arguments to predict which particle has the highest probability of succeeding. (b) Evaluate quantitatively the probability of success for both particles.

<解>：(a) The opacity of a barrier is proportional to $\frac{2mV_0a^2}{\hbar^2}$ and therefore the lower mass particle (proton) has the higher probability of getting through.

(b) With $V_0 = 10MeV$, $E = 3MeV$, $a = 10^{-14}m$, it follows that

$$16\frac{E}{V_0}(1-\frac{E}{V_0}) = 3.36.$$

The required masses are $m_p = 1.673\times10^{-27}kg$, $m_d \approx 2m_p$. For the proton $k_2a = 5.803$ and, using the approximate formula,

$$T_p = 3.36e^{-2(5.083)} = 3.06\times10^{-5}.$$

Since $m_d \approx 2m_p$, as noted above, $k_2a \simeq \sqrt{2}\times5.803 = 8.207$. Hence, for the deuteron, $T_d = 3.36e^{-2(8.207)} = 2.5\times10^{-7}$ ……##

6-10、A fusion reaction important in solar energy production (see Question 16) involves capture of a proton by a carbon nucleus, which has six times the charge of a proton and a radius of $r' \simeq 2\times10^{-15}m$. (a) Estimate the Coulomb potential V experienced by the proton if it is at the nuclear surface. (b) The proton is incident upon the nucleus because of its thermal motion. Its total energy cannot realistically be assumed to be much higher than 10kT, where k is Boltzmann's constant (see Chapter 1) and where T is the internal temperature of the sun of about $10^{7\,0}K$. Estimate this total energy, and compare it with the height of Coulomb barrier. (c) Calculate the probability that the proton can penetrate a rectangular barrier potential of height V extending from r' to r'', the point at which the Coulomb barrier potential drops to $\frac{V}{2}$. (d) IS the penetration through the actual Coulomb barrier potential greater or less than through the rectangular barrier potential of part (c)?

<解>：(a) $V_0 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{r'} = (9\times10^9)\dfrac{(6)(1)(1.6\times10^{-19})^2}{2\times10^{-15}}$

$$V_0 = \frac{6.912\times10^{-13}\,J}{1.6\times10^{-13}\,J/MeV} = 4.32MeV$$

(b) $E = 10kT = (10)(1.38\times10^{-23})(10^7) = 1.38\times10^{-15}\,J$

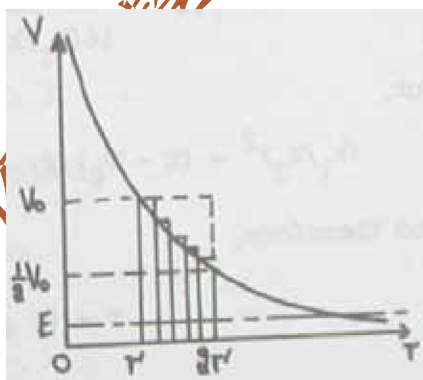
$$= 8.625\times10^{-3}\,MeV = 0.002V_0$$

(c) Numerically, $a = 2r' - r' = 2\times10^{-15}\,m$;

also, $16\dfrac{E}{V_0}(1-\dfrac{E}{V_0}) = 0.032$; $k_2 a = \dfrac{\sqrt{2m(V_0-E)}}{\hbar}a = 0.91$

$$T = \{1 + \frac{(2.484-0.403)^2}{0.032}\}^{-1} = 0.0073$$

(d) The actual barrier can be considered as a series of barriers, each of constant height but the heights decreasing with r; hence $V_0 - E$ diminishes with r and the probability of penetration is greater than for an equal width barrier of constant height V_0……##



<註>：課本 Appendix S 答案：(10a) $4.32MeV$ (10b) $2\times10^{-3}V_0$ (10c) 0.0073

6-11、Verify by substitution that the standing wave general solution, (6-62), satisfies the time- independent Schroedinger equation, (6-2), for the finite square well potential in the region inside the well.

<解>：

6-12、Verify by substitution that the exponential general solutions, (6-63) and (6-64), satisfy the time- independent Schroedinger equation (6-13) for the finite square well potential in the regions outside the well.

<解>：

6-13、(a) From qualitative arguments, make a sketch of the form of a typical unbound standing wave eigenfunction for a finite square well potential. (b) Is the amplitude of the oscillation the same in all regions? (c) What does the behavior of the amplitude predict about the probabilities of finding the particle in a unit length of the x axis in various regions? (d) Does the prediction agree with what would be expected from classical mechanics?

<解>：

6-14、Use the qualitative arguments of Problem 13 to develop a condition on the total energy of the particle, in an unbound state of a finite square well potential, which makes the probability of finding it in a unit length of the x axis the same inside the well as outside the well. (Hint : What counts is the relation between the de Broglie wavelength inside the well and the width of the well.)

<解>：

6-15、(a) Make a quantitative calculation of the transmission coefficient for an unbound particle moving over a finite square well potential. (Hint : Use a trick similar to the one indicated in Problem 7.) (b) Find a condition on the total energy of the particle which makes the transmission coefficient equal to one. (c) Compare with the condition found in Problem 14, and explain why they are the same. (d) Give an example of an optical analogue to this system.

<解>：(a) $[\frac{1+(\sin^2 k_2 a)}{4x(x-1)}]^{-1}$, $x=\frac{E}{V_0}$

(b) $\frac{n^2\pi^2\hbar^2}{2ma^2}$

(c)

(d)

6-16、(a) Consider a one-dimensional square well potential of finite depth V_0 and width a. What combination of these parameters determines the "strength" of the well-i.e., the number of energy levels the wells is capable of binding? In the limit that the strength of the well becomes small, will the number of bound levels become 1 or 0? Give convincing justification for your answers.

<解>：

6-17、An atom of noble gas krypton exerts an attractive potential on an unbound electron, which has a very abrupt onset. Because of this it is a reasonable approximation to describe the potential as an attractive square well, of radius equal to the $4\times10^{-10}m$ radius of the atom. Experiments show that an electron of kinetic energy 0.7eV, in regions outside the atom, can travel through the atom with essentially no reflection. The phenomenon is called the Ramsaure effect. Use this information in the conditions of Problem 14 or 15 to determine the depth of the square well potential. (Hint：One de Broglie wavelength just fits into the width of the well. Why not one-half a de Broglie wavelength?)

<解>：Numerically $a=2(4\times10^{-10}m)$ and $K=0.7eV$. $E=K+V_0$ where

$$E=\frac{n^2h^2}{8ma^2}=n^2\frac{(6.626\times10^{-34})^2}{8(9.11\times10^{-31})(8\times10^{-10})^2(1.6\times10^{-19})}=n^2(0.588eV)$$

Set $n=1$; $E_1=0.588eV<K$, which is not possible.

Using $n=2$ gives $E_2=2^2E_1=2.352eV$

$$V_0=E-K=1.65eV$$

The electron is too energetic for only half its wavelength to fit into the well; this may be verified by calculating the deBroglie wavelength of an electron with a kinetic energy over the well of 2.35eV……##

6-18、A particle of total energy $9V_0$ is incident from the $-x$ axis on a potential given by $V = \begin{cases} 8V_0 & x<0 \\ 0 & 0<x<a \\ 5V_0 & x>a \end{cases}$.

Find the probability that the particle will be transmitted on through to the positive side of the *x* axis, $x>a$.

<解>：

6-19、Verify by substitution that the standing wave general solution, (6-67), satisfies the time-independent Schroedinger equation (6-2), for the infinite square well potential in the region inside the well.

<解>：

6-20、Two possible eigenfunctions for a particle moving freely in a region of length *a*, but strictly confined to that region, are shown in Figure 6-37. When the particle is in the state corresponding to the eigenfunction ψ_I, its total energy is 4eV. (a) What is its total energy in the state corresponding to ψ_{II}? (b) What is the lowest possible total energy for the particle in this system?

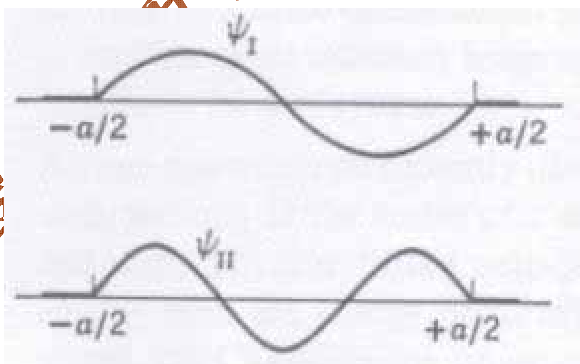


Figure 6-37 Two eigenfunctions considered in Problem 20

<解>：(a) In the lowest energy state $n=1$, ψ has no nodes. Hence ψ_I must correspond to $n=2$, ψ_{II} to $n=3$. Since $E_n \propto n^2$ and

$$E_I = 4eV, \frac{E_{II}}{E_I} = \frac{3^2}{2^2};\ E_{II} = 9eV.$$

(b) By the same analysis, $\frac{E_0}{E_I} = \frac{1^2}{2^2}$; $E_0 = 1eV$ …….##

6-21、(a) Estimate the zero-point energy for a neutron in a nucleus, by treating it as if it were in an infinite square well of wide equal to a nuclear diameter of $10^{-14}m$. (b) Compare your answer with the electron zero-point energy of Example 6-6.

<解>：(a) $2.05MeV$

(b)

6-22、(a) Solve the classical wave equation governing the vibrations of a stretched string, for a string fixed at both its ends. Thereby show that functions describing the possible shapes assumed by the string are essentially the same as the eigenfunctions for an infinite square well potential. (b) Also show that the possible frequencies of vibration of the string are essentially different from the frequencies of the wave functions for the potential.

<解>：(a)

(b)

6-23、(a) For a particle in a box, show that the fractional difference in the energy between adjacent eigenvalues is $\dfrac{\Delta E_n}{E_n}=\dfrac{2n+1}{n^2}$. (b) Use this formula to discuss the classical limit of the system.

<解>：(a) The energy in question is $E_n=n^2\dfrac{\pi^2\hbar^2}{2ma^2}$, and therefore the energy of the adjacent level is

$$E_{n+1}=(n+1)^2\frac{\pi^2\hbar^2}{2ma^2},\ \text{so that}\ \frac{\Delta E_n}{E_n}=\frac{E_{n+1}-E_n}{E_n}=\frac{(n+1)^2-n^2}{n^2}=\frac{2n+1}{n^2}.$$

(b) In the classical limit $n\to\infty$; but $\lim\limits_{n\to\infty}\dfrac{\Delta E_n}{E_n}=\lim\limits_{n\to\infty}\dfrac{2n+1}{n^2}=0$

Meaning that the energy levels get so close together as to be indistinguishable. Hence, quantum effects are not apparent.

6-24、Apply the normalization condition to show that the value of the multiplicative constant for the $n=3$ eigenfunction of the infinite square well potential, (6-79), is $B_3=\sqrt{\frac{2}{a}}$.

<解>：The eigenfunctions for odd n are $\psi_n = B_n\cos\frac{n\pi x}{a}$.

For normalization, $1=\int\psi_n^2dx = B_n^2\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos^2\frac{n\pi x}{a}dx = 2B_n^2\frac{a}{n\pi}\int_0^{\frac{n\pi}{2}}\cos^2 u du$

$1=2B_n^2(\frac{a}{n\pi})(\frac{n\pi}{4})=\frac{a}{2}B_n^2 \Rightarrow B_n=\sqrt{\frac{2}{n}}$

For all odd n and, therefore, for $n=3$.

6-25、Use the eigenfunction of Problem 24 to calculate the following expectation values, and comment on each result : (a) $\bar{x}$, (b) $\bar{p}$, (c) $\overline{x^2}$, (d) $\overline{p^2}$.

<解>：(a) zero

(b) zero

(c) $0.0777a^2$

(d) $88.826(\frac{\hbar}{a})^2$

6-26、(a) Use the results of Problem 25 to evaluate the product of the uncertainly in position times the uncertainty in momentum, for a particle in the $n=3$ state of an infinite square well potential. (b) Compare with the results of Example 5-10 and Problem 13 of Chapter 5, and comment on the relative size of the uncertainty products for the $n=1$, $n=2$, and $n=3$ state. (c) Find the limits of Δx and Δp as n approaches infinity.

<解>：(a) Using the results of the previous problem,

$$\Delta x=\sqrt{\overline{x^2}}=\frac{a}{\sqrt{12}}(1-\frac{6}{n^2\pi^2})^{1/2},\quad \Delta p=\sqrt{\overline{p^2}}=n\pi(\frac{\hbar}{a})$$

Hence, for $n=3$, $\Delta x\Delta p=\frac{a}{\sqrt{12}}(1-\frac{6}{3^2\pi^2})^{1/2}3\pi\frac{\hbar}{a}=2.67\hbar$.

(b) The other results are $n=1$, $\Delta x\Delta p=0.57\hbar$

$n=2$, $\Delta x\Delta p=1.67\hbar$

The increase with n is due mainly to the uncertainty in p: see Problem 6-25.

(c) From (a), the limits as $n \to \infty$ are $\Delta x \to \frac{a}{\sqrt{12}}$; $\Delta p \to \infty$ …….##

6-27、Form the product of the eigenfunction for the $n=1$ state of an infinite square well potential times the eigenfunction for $n=3$ state of that potential. Then integrate it over all x, and show that the result is equal to zero. In other words, prove that $\int_{-\infty}^{\infty}\psi_1(x)\psi_3(x)dx = 0$. (Hint : Use the relation : $\cos u \cos v = \frac{\cos(u+v)+\cos(u-v)}{2}$.) Students who have worked Problem 36 of Chapter 5 have already proved that the integral over all x of the $n=1$ eigenfunction times the $n=2$ eigenfunction also equals zero. It can be proved that the integral over all x of any two different eigenfunctions of the potential equals zero. Furtherrmore, this is true for any two different eigenfunctions of any other potential. (If the eigenfunctions are complex, the complex conjugate of one is taken in the integrand.) This property is called orthogonality.

<解>：$\int_{-\infty}^{+\infty}\psi_1\psi_3 dx = \frac{2}{a}\int_{-\frac{a}{2}}^{+\frac{a}{2}}\cos\frac{\pi x}{a}\cos\frac{3\pi x}{a}dx = \frac{1}{a}\int_{-\frac{a}{2}}^{+\frac{a}{2}}\{\cos\frac{4\pi x}{a} - \cos\frac{2\pi x}{a}\}dx$

$$\int_{-\infty}^{+\infty}\psi_1\psi_3 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos 2u - \cos u)du$$

The integrand being an even function of u…….##

6-28、Apply the results of Problem 20 of Chapter 5 to the case of a particle in a three-dimensional box. That is, solve the time-independent Schroedinger equation for a particle moving in a three-dimensional potential that is zero inside a cubical region of edge length *a*, and becomes infinitely large outside that region. Determine the eigenvalues and eigenfunctions for system.

<解>：

6-29、Airline passengers frequently observe the wingtips of their planes oscillating up and down with periods of the order of 1 sec and amplitudes of about 0.1m. (a) Prove that this is definitely not due to the zero-point motion of the wings by comparing the zero-point energy with the energy obtained from the quoted values plus an estimated mass for the wings. (b) Calculate the order of magnitude of the quantum number *n* of the observed oscillation.

<解>：(a) Let $M = mass$ of wing. The zero-point energy is

$$E_0 = (1+\frac{1}{2})\hbar\omega = \frac{1}{2}h\nu = \frac{h}{2T},$$

$T =$ period of oscillation. The actual energy of oscillation is

$$E = \frac{1}{2}kA^2 = \frac{1}{2}M\omega^2 A^2 = \frac{2\pi^2 MA^2}{T^2}$$

Thus, the value of M at which $E = E_0$ is

$$M = \frac{hT}{4\pi^2 A^2} = \frac{(6.626\times10^{-34})(1)}{4\pi^2(10^{-1})^2} = 1.68\times10^{-33}kg$$

This is less than the mass of an electron. Hence $E >> E_0$ and the observed vibration is not the zero-point motion.

(b) Clearly then, $n >> 1$ and therefore $E = nh\nu = \frac{2\pi^2 MA^2}{T^2} \to n = \frac{2\pi^2 MA^2}{hT}$

As an example, take $M = 2000kg$: $n = \frac{2\pi^2(2000)(10^{-1})^2}{(6.626\times10^{-34})(1)} = 6\times10^{35}$ ....##

<註>：課本 Appendix S 答案：(29b) $\simeq 10^{36}$

6-30、The restoring force constant C for the vibrations of the interatomic spacing of a typical diatomic molecule is about $10^3\ joule/m^2$. Use this value to estimate the zero-point energy of the molecular vibrations. The mass of the molecule is $4.1\times10^{-26}kg$.

<解>：The zero-point energy is $E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar(\frac{C}{m})^{1/2}$

Therefore, $E_0 = \frac{1}{2}(1.055\times10^{-34})(\frac{10^3}{4.1\times10^{-26}})^{1/2}(1.6\times10^{-19})^{-1}$

$E_0 = 0.051eV$ ......##

6-31、(a) Estimate the difference in energy between the ground state and first excited state of the vibrating molecule considered in Problem 30. (b) From this estimate determine the energy of the photon emitted by the vibrations in the charge distribution when the system makes a transition between the first excited state and the ground state. (c) Determine also the frequency of the photon, and compare it with the classical oscillation frequency of the system. (d) In what range of the electromagnetic spectrum is it?

<解>：(a) Using $E_0 = 0.051eV$, the level spacing will be

$$\Delta E = \Delta(n+\frac{1}{2})\hbar\omega = \hbar\omega = 0.102eV = 2E_0.$$

(b) The energy E of the proton $= \Delta E = 0.102eV$.

(c) For the proton, $E = \hbar\omega_{ph}$

But $E = \Delta E = \hbar\omega \Rightarrow \omega_{ph} = \omega$

Where $\omega =$ classical oscillation frequency. Thus,

$$\nu = \frac{E}{h} = \frac{(0.102)(1.6\times10^{-19})}{6.626\times10^{-34}} = 2.5\times10^{13}Hz.$$

(d) Photons of this frequency are in the infrared spectrum,

$\lambda = 12,000nm$ ....##

6-32、A pendulum, consisting of a weight of 1kg at the end of a light 1m rod, is oscillating with an amplitude of 0.1m. Evaluate the following quantities : (a) Frequency of oscillation, (b) energy of oscillation, (c) approximate value of quantum number for oscillation, (d) separation in energy between adjacent allowed energies, (e) separation in distance between adjacent bumps in the probability density function near the equilibrium point.

<解>：(a) $\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{1}} = 3.13rad/s \Rightarrow \nu = \frac{\omega}{2\pi} = 0.498Hz$.

(b) $E = \frac{1}{2}kA^2 = \frac{1}{2}\frac{mg}{L}A^2 \Rightarrow E = 0.049J$.

(c) Since $n >> 1$, $n = \frac{E}{h\nu} = \frac{0.0490}{(6.626\times10^{-34})(0.498)} = 1.5\times10^{32}$.

(d) Since $\Delta n = 1$, $\Delta E = h\nu = 3.3\times10^{-34}J$.

(e) A polynomial of degree n has n nodes; hence, the distance between "bumps"=distance adjacent

$$nodes = \frac{2A}{n} = \frac{2(0.1)}{1.5\times10^{32}} = 1.3\times10^{-33}\,m \text{......##}$$

<註>：課本 Appendix S 答案：(32a) $0.5Hz$ (32b) $0.049\,joule$ (32c) 1.5×10^{32} (32d) $3.3\times10^{-34}\,joule$ (32e) $1.3\times10^{-33}\,m$

6-33、Devise a simple argument verifying that the exponent in the decreasing exponential, which governs the behavior of simple harmonic oscillator eigenfunctions in the classically excluded region, is proportional to x^2. (Hint : Take the finite square well eigenfunctions of (6-63) and (6-64), and treat the quantity $(V_0 - E)$ as if it increased with increasing x in proportion to x^2.)

<解>：

6-34、Verify the eigenfunction and eigenvalue for the $n = 2$ state of a simple harmonic oscillator by direct substitution into the time-independent Schroedinger equation, as in Example 6-7.

<解>：
