Quantum Physics（量子物理）習題
Robert Eisberg（Second edition）
CH 11：Quantum statistics

11-01、The equilibrium state is one of maximum entropy S in thermodynamics and one of maximum probability P in statistics. Assuming then that S is a function of P, show that we should expect $S = k\ln P$, where k is a universal constant. This relation is sometimes called the Boltzmann postulate. (Hint : Consider the effect on S and P of combining two systems.)

ANS：

11-02、The Maxwell distribution can be developed by looking at elastic collisions between two particles. If initially these particles have energies ε_1 and ε_2, and finally ε_3 and ε_4, then $\varepsilon_3 + \varepsilon_4 = (\varepsilon_1 - \delta) + (\varepsilon_2 - \delta)$ If all possible states are equally probable , the number of collisions per second P is proportional to the number of particles in each initial state, i.e. $P_{1,2} = CP(\varepsilon_1)P(\varepsilon_2)$, where $P(\varepsilon_i)$ is the probability of a state being occupied, and C is a constant. Similarly $P_{3,4} = CP(\varepsilon_3)P(\varepsilon_3)$. In equilibrium, for each collision $(1,2) \to (3,4)$ there must be a collision $(3,4) \to (1,2)$. Thus $P_{1,2} = P_{3,4}$. (a) Show that $P(\varepsilon_i) = e^{-\frac{\varepsilon_i}{kT}}$ solves this equation. (b) Use similar reasoning to derive the Fermi distribution. Here, however, the initial states must be filled and the final states must be empty, and the number of collisions becomes $P_{1,2} = CP(\varepsilon_1)P(\varepsilon_2)[1 - P(\varepsilon_3)][1 - P(\varepsilon_4)]$. Then show that the equation $P_{1,2} = P_{3,4}$ can be solved by $[\dfrac{1 - P(\varepsilon_i)}{P(\varepsilon_i)}] = Ce^{\frac{\varepsilon_i}{kT}}$ which yields (11-23).

ANS：

11-03、(a) Show that at $T = 0$, in the Fermi distribution, $n(\varepsilon) = 1$ for all energy states in which $\varepsilon \le \varepsilon_F$ and $n(\varepsilon) = 0$ for all energy states in which $\varepsilon > \varepsilon_F$. (b) Show that $n(\varepsilon) = \dfrac{1}{2}$ for $\varepsilon = \varepsilon_F$.

ANS：

11-04、Consider the Fermi distribution of (11-24), $n(\varepsilon)=\dfrac{1}{e^{\frac{\varepsilon-\varepsilon_F}{kT}}+1}$. (a) Show that $n(\varepsilon)=1-n(2\varepsilon_F-\varepsilon)$; that is, with $\varepsilon_F-\varepsilon=\delta$, show that $n(\varepsilon_F+\delta)=1-n(\varepsilon_F-\delta)$. This proves that the distribution has a symmetry about $n(\varepsilon_F)=\dfrac{1}{2}$. (b) Find $n(\varepsilon)$ for $\delta=\varepsilon-\varepsilon_F=kT$, or $2kT$, or $4kT$, or $10kT$. Make a rough sketch of $n(\varepsilon)$ versus ε for any $T>0$. (c) What percent error is made by approximating the Fermi distribution by the Boltzmann distribution when $\dfrac{\delta}{kT}=1,2,4,10$?

ANS：

11-05、(a) At what energy is the Bose distribution function (for $\alpha=0$) equal to one for a temperature of 7000^0K ? (b) What is the temperature of the Bose function (for $\alpha=0$) with a value of 0.500 at this same energy?

ANS：(a) $0.418eV$ (b) 4410^0K

11-06、For the Fermi distribution function (a) show that $\int_0^{\varepsilon_F} n(\varepsilon)d\varepsilon=kT[\ln\dfrac{1+e^{\frac{\varepsilon_F}{kT}}}{2}]$. (b) Show that this reduces to ε_F for $T=0$. (c) Show that $\int_0^{\infty} n(\varepsilon)d\varepsilon=\int_0^{\varepsilon_F} n(\varepsilon)d\varepsilon+kT(\ln 2)$.

ANS：

11-07、(a) From (11-25), show that the Einstein model of a solid gives the specific heat as $c_v=3R[\dfrac{e^{\frac{h\nu}{kT}}}{(e^{\frac{h\nu}{kT}}-1)^2}(\dfrac{h\nu}{kT})^2]$. (b) Show that $c_v\to 0$ as $T\to 0$ but that at low T, c_v increases as $e^{-\frac{h\nu}{kT}}$ rather than as the require T^3 law.

ANS：

11-08、Show that the Debye specific heat result, (11-31), reduces to the classical law of Dulong and Petit at high temperatures. (Hint : First expand both exponentials and retain only first order terms. Justify.)

ANS：

11-09、Imagine a cavity at temperature *T*. Show that c_v, the specific heat of the enclosed radiation, is given by $\frac{32\pi^5 kV}{15}(\frac{kT}{hc})^3$. Explain why c_v does not have an upper limit in this case whereas it does for solids.

ANS：

11-10、In some temperature region graphite can be considered a two-dimensional Debye solid, but there are still $3N_0$ modes per mole. (a) Show that $N(\nu)d\nu = \frac{2\pi A}{v^2}\nu d\nu$ where *A* is the area of the sample. (b) Find an expression for ν_m and Θ for graphite. (c) Show that at low temperatures the heat capacity is proportional to T^2.

ANS：(b) $\nu_m = v\sqrt{\frac{3N_0}{\pi a}}$, $\theta = \frac{h\nu}{k}\sqrt{\frac{3N_0}{\pi a}}$

11-11、*N* distinguishable atom are distributed over two energy levels $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon$. (a) Show that the energy of the system is given by $E = \frac{N\varepsilon e^{-\frac{\varepsilon}{kT}}}{1+e^{-\frac{\varepsilon}{kT}}}$. (b) Show that c_v is given by $c_v = \frac{Nk(\frac{\varepsilon}{kT})^2 e^{-\frac{\varepsilon}{kT}}}{(1+e^{-\frac{\varepsilon}{kT}})^2}$. (Hint : This is the Schottky specific heat and is observed for paramagnetic solids at low temperature. The energy levels correspond to the magnetic moments being aligned parallel or antiparallel to the magnetic field.) (c) Sketch the heat capacity as a function of temperature, being careful to have the correct temperature dependence at high and low temperatures.

ANS：

11-12、the variation of density ρ with altitude *y* of the gaseous atmosphere of the earth can be written as $\rho = \rho_0 e^{-g(\frac{\rho_0}{P_0})y}$, where ρ_0 and P_0 are sea level density and pressure, provided the temperature is assumed to be uniform. (a) From the ideal gas laws show that this can be put into the form $\rho = \rho_0 e^{\frac{mgy}{kT}}$. (b) Show that this has the form of the Boltzmann distribution.

ANS：

11-13、(a) By combining $n(\varepsilon)$ of (11-21) and $N(\varepsilon)$ of (11-49) for an ideal gas of classical particles, with $A = e^{-\alpha} = \dfrac{Nh^3}{(2\pi mkT)^{3/2}V}$, show that $n(\varepsilon)N(\varepsilon)d\varepsilon = \dfrac{2N}{(kT)^{3/2}\pi^{1/2}}\varepsilon^{1/2}e^{-\frac{\varepsilon}{kT}}d\varepsilon$ is the energy distribution of particles in an ideal gas. (b) Show that Maxwell's speed distribution of molecules in a gas, which has the form $n(v)dv = Cv^2e^{-\frac{mv^2}{2kT}}dv$, where C is a constant, follows directly from this.

ANS：

11-14、Assume that the thermal neutrons emerging from a nuclear reactor have an energy distribution corresponding to a classical ideal gas at a temperature of $300\,^0K$. Calculate the density of neutrons in a beam of flux $10^{13}/m^2-\text{sec}$. (Hint : Consider the average velocity, and justify its use.)

ANS：

11-15、In a certain nucleus the magnetic moment is $1.4\times10^{-26}\,joule-m^2/weber$. Calculate the fractional difference in population of the nuclear Zeeman levels in a magnetic field of $1weber/m^2$, (a) at room temperature and (b) at $4\,^0K$.

ANS：

11-16、Electron spin resonance is much like nuclear magnetic resonance expect that electronic transitions are excited between atomic Zeeman levels. These experiments are done at microwave frequencies. If the electromagnetic wave has a frequency of $32KMHz$ (K band) calculate the fractional difference in population between two atomic Zeeman levels (a) at room temperature and (b) at $4\,^0K$.

ANS：

11-17、(a) Determine the order of magnitude of the fraction of hydrogen atoms in a state with principle quantum number $n=2$ to those in state $n=1$ in a gas at $300\,^0K$. (b) Take into account the degeneracy of the states corresponding to quantum numbers $n=1$ and 2 of atomic hydrogen and determine at what temperature approximately one atom in a hundred is in a state with $n=2$.

ANS：

11-18、Consider the relation $\frac{n_1}{n_2}=e^{\frac{\varepsilon_2-\varepsilon_1}{kT}}$, the Boltzmann factor nondegenerate states for systems in equilibrium, where $\varepsilon_2>\varepsilon_1$. (a) Show that $n_2=0$ at $T=0$. (b) Show that $n_1=n_2$ at $T=\infty$ or $T=-\infty$. (c) Show that $n_2>n_1$ at finite negative temperature *T*. (d) Show that $n_1\to 0$ as $T\to -0$. (e) Hence, explain the statements, "Negative absolute temperatures are not colder than absolute zero but hotter than infinite temperature," and "One approaches negative temperatures through infinity, not through zero." (f) Can you suggest a change in temperature scale that would avoid temperatures that are negative in this sense?

ANS：

11-19、Determine approximately the ratio of the probability of spontaneous emission to the probability of stimulated emission at room temperature in (a) the x-ray region of the electromagnetic spectrum, (b) the visible region, (c) the microwave region.

ANS：

11-20、An atom has two energy levels with a transition wavelength of 5800Å. At room temperature 4×10^{20} atoms are in the lower state. (a) How many occupy the upper state, under conditions of thermal equilibrium? (b) Suppose instead that 7×10^{20} atoms are pumped into the upper state, with 4×10^{20} in the lower state. How much energy in joules could be released in a single pulse?

ANS：(a) none (b) 51.4 *joule*

11-21、The energy levels in a two-level atom are separated by 2.00eV. There are 3×10^{8} atoms in the upper level and 1.7×10^{8} atoms in the ground level. The coefficient of stimulated emission is $3.2\times10^{5}m^{3}/W-\sec^{3}$, and the spectral radiancy is $4W/m^{2}-Hz$. Calculate the stimulated emission rate.

ANS：$1.28\times10^{16}\sec^{-1}$

11-22、If $B_{10}=2.7\times10^{19}m^{3}/W-\sec^{3}$ for a particular atom, find the life-time of the 1 to 0 transition at (a) 5500Å (visible) and (b) 550Å (ultraviloet)?

ANS：

11-23、Combine (11-49) and (11-47) to obtain (11-50), as follows. Let $x=\frac{\varepsilon}{kT}$ and obtain $N=\frac{2\pi V(2mkT)^{3/2}}{h^3}\int_0^\infty \frac{x^{1/2}dx}{e^{\alpha+x}-1}$. Then, with α positive, use the relation $(e^{\alpha+x}-1)^{-1}=e^{-\alpha-x}(1-e^{-\alpha-x})^{-1}=e^{-\alpha}(e^{-x}+e^{-\alpha-2x}+...)$ to obtain (11-50).

ANS：

11-24、Obtain (11-52) as follows. Let $x=\frac{\varepsilon}{kT}$ and show that

$$E=\frac{2\pi kTV(2mkT)^{3/2}}{h^3}\int_0^\infty \frac{x^{3/2}dx}{e^{\alpha+x}-1}=\frac{3}{2}kT\frac{V(2\pi mkT)^{3/2}}{h^3}e^{-\alpha}(1+\frac{1}{2^{5/2}}e^{-\alpha}+...)$$

ANS：

11-25、Show that the quantum degeneracy in a Fermi gas occurs if $kT<<\varepsilon_F$. (Hint : See Example 11-4 and use (11-57).)

ANS：

11-26、Show from the Fermi distribution that in a metal at $T=0^0K$ the average energy of an electron is $\frac{3\varepsilon_F}{5}$.

ANS：

11-27、Using 23 as the atomic weight and $9.7\times10^2 kg/m^3$ as the density of metallic sodium, compute the Fermi energy on the assumption that each sodium atom gives one electron to the conduction band. (Hint : See Example 11-5.)

ANS：$3.1eV$

11-28、Using 197 as the atomic weight and $19.3\times10^3 kg/m^3$ as the density of gold, compute the depth of the potential well for free electrons in gold. The work function is 4.8eV and there is one free electron per atom.

ANS：$10.3eV$

11-29、In a one-dimensional system the number of energy states per unit energy is $\frac{l}{h}\sqrt{\frac{2m}{\varepsilon}}$, where l is the length of the sample and m is the mass of the electron. There are N electrons in the sample and each state can be occupied by two electrons. (a) Determine the Fermi energy at 0^0K. (b) Find the average energy per electron at 0^0K.

ANS：(a) $\frac{N^2h^2}{32ml^2}$ (b) $\frac{\varepsilon_F}{3}$

11-30、Show that about one conduction electron in a thousand in metallic silver has an energy greater than the Fermi energy at room temperature.

ANS：
