積分表證明

100/08/19（六）整理更新

【1】$\int udv = uv - \int vdu$

<解>：$d(uv) = u \cdot dv + du \cdot v$ ……（Product Rule）

$$uv = \int udv + \int vdu$$

$$\int udv = uv - \int vdu$$

（$\int w'vdu = wv - \int wv'du$ ）

【2】$\int u^2 du = \frac{u^{n+1}}{n+1} + C$ ，$n \neq -1$

<解>：

【3】$\int \frac{du}{u} = \ln|u| + C$

<解>：

【4】$\int e^u du = e^u + C$

<解>：

【5】$\int a^u du = \frac{a^u}{\ln a} + C$

<解>：

【6】$\int \sin u du = -\cos u + C$

<解>：

【7】$\int \cos u du = \sin u + C$

<解>：

【8】$\int \sec^2 u du \ = \ \tan u + C$

<解>：$\int \frac{1}{\cos^2 u} du = \int \frac{2}{1+\cos 2u} du$ ……(1)

令 $t = \tan u \Rightarrow dt = \frac{1}{\cos^2 u} du = (1+t^2) du$

則 $\cos 2u = \frac{1-t^2}{1+t^2}$

$\sin 2u = \frac{2t}{1+t^2}$

代入(1)式 $\Rightarrow \int \frac{2}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{dt}{1+t^2} = \int 1 dt$

$= t + c$

$= \tan u + c$

【9】$\int \csc^2 u du \ = -\cot u + C$

<解>：$\int \frac{1}{\sin^2 u} du = \int \frac{2}{1-\cos 2u} du$ ……(1)

令 $t = \tan u \Rightarrow dt = \frac{1}{\cos^2 u} du = (1+t^2) du$

則 $\cos 2u = \frac{1-t^2}{1+t^2}$

$\sin 2u = \frac{2t}{1+t^2}$

代入(1)式 $\Rightarrow \int \dfrac{2}{1-\dfrac{1-t^2}{1+t^2}}\cdot\dfrac{dt}{1+t^2}=\int\dfrac{1}{t^2}dt$

$=-t^{-1}+C$

$=-\cot u+C$

【10】$\int \sec u\tan u du=\sec u+C$

<解>：原式$=\int\dfrac{\sin u}{\cos^2 u}du$

方法一：令 $t=\cos u \Rightarrow dt=-\sin u du$

則原式 $\Rightarrow \int -\dfrac{1}{t^2}dt=t^{-1}+C=\sec u+C$

方法二：原式相當習題 2.42 $J(-2,1)$

$\int w'vdu=wv-\int wv'du$

令 $w'=\cos^{-2}u\sin u \qquad w=(\cos u)^{-1}$

$v=1 \qquad v'=0$

$\int\dfrac{\sin u}{\cos^2 u}du=(\cos u)^{-1}(1)-\int(\cos u)(0)du$

$=\sec u+C$

若令 $\begin{pmatrix} w'=\sin u\cos u & w=\dfrac{1}{2}\sin^2 u \\ v=\cos^{-3}u & v'=-3\cos^{-2}u \end{pmatrix}$

則題目會變得更複雜

【11】$\int\csc u\cot u du=-\csc u+C$

<解>：原式 $=\int\dfrac{\cos u}{\sin^2 u}du$ ……(1)

方法一：令 $t=\sin u$，$dt=\cos u du$

代入(1) $\int\dfrac{1}{t^2}dt=-t^{-1}+C=-\csc u+C$

方法二：原式相當習題 2.42 $J(1,-2)$

$\int w'vdu=wv-\int wv'du$

令 $w'=\sin^{-2}u\cos u \qquad w=-(\sin u)^{-1}$

$v=1 \qquad v'=0$

$\int\dfrac{\cos u}{\sin^2 u}du=-(\sin u)^{-1}(1)-\int-(\sin u)(0)du$

$=\csc u+C$

【12】$\int\tan u du=\ln|\sec u|+C$

<解>：原式$=\int\dfrac{\sin u}{\cos u}du$

令 $t=\cos u \Rightarrow dt=-\sin u du$

代入上式 $\Rightarrow \int-\dfrac{1}{t}dt=-\ln|t|+C$

$=\ln\dfrac{1}{|t|}+C$

$=\ln|\sec u|+C$

【13】$\int\cot u du=\ln|\sin u|+C$

<解>：原式$=\int\dfrac{\cos u}{\sin u}du$ ……(1)

令 $t=\sin u$，$dt=\cos u du$

代入(1) $\Rightarrow \int\dfrac{dt}{t}=\ln|t|+C$

$=\ln|\sin u|+C$

【14】$\int\sec u du=\ln|\sec u+\tan u|+C$

<解>：原式$=\int\dfrac{1}{\cos u}du$ ……(1)

令 $t=\tan\dfrac{u}{2} \Rightarrow dt=\dfrac{1}{\cos^2\dfrac{u}{2}}\cdot\dfrac{1}{2}du=(1+t^2)\cdot\dfrac{1}{2}du$ 代入(1)

$$\begin{pmatrix} \cos u &= \dfrac{1-t^2}{1+t^2} \\ \sec u &= \dfrac{1+t^2}{1-t^2} \\ \tan u &= \dfrac{2t}{1-t^2} \end{pmatrix}$$

$$\Rightarrow \int \frac{1+t^2}{1-t^2}\cdot\frac{2}{1+t^2}dt = \int \frac{2}{1-t^2}dt$$

$$= \int \frac{1}{1-t}+\frac{1}{1+t}dt$$

$$= -\ln|1-t| + \ln|1+t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln|\sec u + \tan u| + C$$

$$\left(\frac{1+t}{1-t}\cdot\frac{1+t}{1+t} = \frac{(1+t)^2}{1-t^2} = \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} = \sec u + \tan u \right)$$

【15】 $\int \csc u du = \ln|\csc u - \cot u| + C$

<解>：原式 $= \int \frac{1}{\sin u} du$ ……(1)

令 $t = \tan\frac{u}{2}$，$dt = (1+t^2)\cdot\frac{1}{2}du$

$\sin u = \frac{2t}{1+t^2}$ 代入(1)

$$\Rightarrow \int \frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}dt = \int \frac{1}{t}dt$$

$$= \ln|t| + C$$

$$= \ln|\csc u - \cot u| + C$$

$$\left(\csc u - \cot u = \frac{1+t^2}{2t} - \frac{1-t^2}{2t} = \frac{2t^2}{2t} = t \right)$$

【16】 $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

<解>：令 $u = a\sin x$，$du = a\cos x dx$ 代入

$$\Rightarrow \int \frac{1}{a\cos x} a\cos x dx = x + C$$

$$= \sin^{-1}\frac{u}{a} + C$$

【17】 $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

<解>：令 $u = a\tan x$，$du = \frac{a}{\cos^2 x}dx$ 代入

$$\Rightarrow \int \frac{1}{a^2\sec^2 x}\cdot\frac{a}{\cos^2 x}dx = \frac{1}{a}\int 1dx$$

$$= \frac{x}{a} + C$$

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

【18】 $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

<解>：令 $u = a\sec x$，$du = a\frac{\sin x}{\cos^2 x}dx$ 代入

$$\Rightarrow \int \frac{1}{a\sec x\cdot a\tan x}\cdot a\cdot\frac{\sin x}{\cos^2 x}dx = \frac{1}{a}\int dx$$

$$= \frac{1}{a}x + C$$

$$= \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

【19】 $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

<解>：$\int \frac{du}{a^2-u^2} = -\int \frac{1}{u^2-a^2}du$

$$= -\frac{1}{2a}\int -\frac{1}{u+a} + \frac{1}{u-a}du$$

$$= \frac{1}{2a}(\ln|u+a| - \ln|u-a|) + C$$

$$= \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

【20】$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$

<解>：$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\int\left(-\frac{1}{u+a}+\frac{1}{u-a}\right)du$

$$=\frac{1}{2a}\left(\ln|u-a|-\ln|u+a|\right)+C$$

$$=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$$

【提示】21、22 解題技巧

雙曲三角函數公式

$\cosh^2 x = 1+\sinh^2 x$

$\sinh x = \frac{e^x-e^{-x}}{2}$，$(\sinh x)' = \cosh x$

$\cosh x = \frac{e^x+e^{-x}}{2}$，$(\cosh x)' = \sinh x$

$t=\sinh x \Rightarrow x=\sinh^{-1} t=\ln\left(t+\sqrt{t^2+1}\right)$

$t=\cosh x \Rightarrow x\ \cosh^{-1} t=\ln\left(t+\sqrt{t^2-1}\right)$

$\sinh x = \frac{e^x-e^{-x}}{2}=\frac{u}{a}$

$e^{2x}-\frac{2u}{a}e^x-1=0 \Rightarrow ae^{2x}-2ue^x-a=0$

$e^x = \frac{2u\pm\sqrt{4u^2+4a^2}}{2a}=\frac{u\pm\sqrt{u^2+a^2}}{a}$（取正）

$x=\ln\frac{u+\sqrt{u^2+a^2}}{a}$

此外 $\cosh x=\sqrt{1+\left(\frac{u}{a}\right)^2}$，$\sinh 2x=\frac{e^{2x}-e^{-2x}}{2}=2\sinh x\cosh x$

【21】$\int\sqrt{a^2+u^2}\,du=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln(u+\sqrt{a^2+u^2})+C$

<解>：令 $u=a\sinh x$，$du=a\cosh x dx$ 代入

$$\int\sqrt{a^2+u^2}\,du=\int\left(\sqrt{a^2+a^2\sinh^2 x}\right)\left(a\cosh x dx\right)$$

$$=\int a\cosh x\cdot a\cosh x dx$$

$$=a^2\int\cosh^2 x dx$$

$$=\frac{a^2}{4}\int e^{2x}+2+e^{-2x}dx$$

$$=\frac{a^2}{4}(\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x})+C$$

$$=\frac{a^2}{4}(\sinh 2x+2x)+C \qquad (\ 2\sinh x\cosh x=2\cdot\frac{u}{a}\cdot\sqrt{1+\left(\frac{u}{a}\right)^2}\ )$$

$$=\frac{a^2}{4}\left(2\cdot\frac{u}{a}\cdot\sqrt{1+\left(\frac{u}{a}\right)^2}+2\ln\frac{u+\sqrt{u^2+a^2}}{a}\right)+C$$

$$=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln(u+\sqrt{u^2+a^2})-\frac{a^2}{2}\ln a+C$$

$$=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln(u+\sqrt{a^2+u^2})+C'$$

【22】$\int u^2\sqrt{a^2+u^2}\,du=\frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2}-\frac{a^4}{8}\ln(u+\sqrt{u^2+a^2})+C$

<解>：令 $u=a\sinh x$，$du=a\cosh x dx$ 代入

$$\int a^2\sinh^2 x\cdot a\cosh x\cdot a\cosh x dx$$

$$=a^4\int\left(\frac{\sinh 2x}{2}\right)^2 dx \qquad (\ \left(\frac{\sinh 2x}{2}\right)^2=\left(\frac{e^{2x}-e^{-2x}}{4}\right)^2\ )$$

$$=\frac{a^4}{16}\int e^{4x}-2+e^{-4x}dx$$

$$=\frac{a^4}{16}\left(\frac{1}{4}e^{4x}-2x-\frac{1}{4}e^{-4x}\right)+C_1 \qquad (\ \frac{1}{4}e^{4x}-\frac{1}{4}e^{-4x}=\frac{\sinh 4x}{2}\ )$$

$$=\frac{a^4}{16}\left(\frac{\sinh 4x}{2}-2x\right)+C_1$$

$$=\frac{a^4}{16}\left[2\sqrt{a^2+u^2}\left(\frac{u}{a^2}+\frac{2u^3}{a^4}\right)-2\ln(u+\sqrt{u^2+a^2})+2\ln a\right]+C_1$$

$$=\frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2}-\frac{a^4}{8}\ln(u+\sqrt{u^2+a^2})+C$$

$$x=\ln\frac{u+\sqrt{u^2+a^2}}{a}=\ln u+\sqrt{u^2+a^2}-\ln a$$

$$\sinh 4x=2\sinh 2x\cosh 2x$$

$$=4\sinh x\cosh x\left(2\cosh^2 x-1\right)$$

$$=4\cdot\frac{u}{a}\cdot\sqrt{1+\left(\frac{u}{a}\right)^2}\left[2(1+\left(\frac{u}{a}\right)^2)-1\right]$$

$$=4\frac{u}{a^2}\sqrt{a^2+u^2}\left(1+\frac{2u^2}{a^2}\right)$$

$$=4\sqrt{a^2+u^2}\left(\frac{u}{a^2}+\frac{2u^3}{a^4}\right)$$

【23】 $\int\frac{\sqrt{a^2+u^2}}{u}du=\sqrt{a^2+u^2}-a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right|+C$

<解>：令 $u=a\tan\theta$，$du=\frac{a}{\cos^2\theta}d\theta$ 代入

$$\sqrt{a^2+u^2}=\frac{a}{\cos\theta}\text{，}\frac{1}{u}=\frac{\cos\theta}{a\sin\theta}$$

$$\int\frac{a}{\cos\theta}\cdot\frac{\cos\theta}{a\sin\theta}\cdot\frac{a}{\cos^2\theta}d\theta=a\int\frac{1}{\sin\theta\cos^2\theta}d\theta\ldots\ldots(1)$$

令 $t=\cos\theta$，$dt=-\sin\theta d\theta$ 代入 (1)

$$-a\int\frac{1}{(1-t^2)(t^2)}dt=-a\int\frac{1}{t^2}+\frac{1}{2(1+t)}+\frac{1}{2(1-t)}dt$$

$$=-a\left[\frac{-1}{t}+\frac{1}{2}\ln 2(t+1)-\frac{1}{2}\ln 2(1-t)\right]+C$$

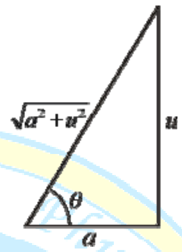
$$=\frac{a}{\cos\theta}-\frac{a}{2}\ln\left|\frac{\cos\theta+1}{1-\cos\theta}\right|+C$$

$$=\sqrt{a^2+u^2}-a\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right|+C$$

$$\frac{\cos\theta+1}{1-\cos\theta}=\frac{\sqrt{a^2+u^2}+a}{\sqrt{a^2+u^2}-a}$$

$$=\frac{\left(\sqrt{a^2+u^2}+a\right)^2}{a^2+u^2-a^2}$$

$$=\left(\frac{\sqrt{a^2+u^2}+a}{u}\right)^2$$



【24】 $\int\frac{\sqrt{a^2+u^2}}{u^2}du=-\frac{\sqrt{a^2+u^2}}{u}+\ln(u+\sqrt{a^2+u^2})+C$

<解>：$u=a\tan x$，$du=\frac{a}{\cos^2 x}dx$ 代入

$$\int\frac{a}{\cos x}\cdot\frac{\cos^2 x}{a^2\sin^2 x}\cdot\frac{a}{\cos^2 x}dx=\int\frac{1}{\sin^2 x\cos x}dx$$ 令 $t=\sin x$，$dt=\cos x dx$

$$=\int\frac{1}{t^2(1-t^2)}dt$$

$$=\int\frac{1}{t^2}+\frac{1}{2(1+t)}+\frac{1}{2(1-t)}dt$$

$$=-\frac{1}{t}+\frac{1}{2}\ln 2(1+t)-\frac{1}{2}\ln 2(1-t)+C'$$

$$=-\frac{1}{t}+\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|+C'$$

$$=-\frac{\sqrt{a^2+u^2}}{u}+\ln\left|\frac{\sqrt{a^2+u^2}+u}{a}\right|+C'$$

$$=-\frac{\sqrt{a^2+u^2}}{u}+\ln\left|\sqrt{a^2+u^2}+u\right|+C$$

$$\frac{1+t}{1-t}=\frac{1+\frac{u}{\sqrt{a^2+u^2}}}{1-\frac{u}{\sqrt{a^2+u^2}}}=\frac{\sqrt{a^2+u^2}+u}{\sqrt{a^2+u^2}-u}=\left(\frac{\sqrt{a^2+u^2}+u}{a}\right)^2$$

【25】 $\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{a^2+u^2})+C$

<解>：令 $u=a\sinh x$，$du=a\cosh x dx$（$x=\ln(u+\sqrt{u^2+a^2})-\ln a$）

$$\int \frac{a\cosh x}{a\cosh x}dx = x+c'$$

$$=\ln(u+\sqrt{u^2+a^2})+C$$

【26】 $\int \frac{u^2}{\sqrt{a^2+u^2}}du = \frac{u}{2}\sqrt{a^2+u^2}-\frac{a^2}{2}\ln(u+\sqrt{a^2+u^2})+C$

<解>：方法一：令 $u=a\sinh x$，$du=a\cosh xdx$

$$\int\frac{a^2(\sinh x)^2 a(\cosh x)dx}{a\cosh x} = a^2\int\sinh^2 x\, dx$$

$$=\frac{a^2}{4}\int e^{2x}-2+e^{-2x}dx$$

$$=\frac{a^2}{4}\left(\frac12 e^{2x}-\frac12 e^{-2x}-2x\right)+C'$$ ……（補充）

$$=\frac{u}{2}\sqrt{a^2+u^2}-\frac{a^2}{2}\ln(u+\sqrt{a^2+u^2})+C$$

（補充）$\frac12 e^{2x}-\frac12 e^{-2x}=\sinh 2x = 2\sinh x\cosh x$

$$=2\left(\frac ua\right)\sqrt{1+\left(\frac ua\right)^2}$$

$$=2\frac{u}{a^2}\sqrt{a^2+u^2}$$

$x=\ln(u+\sqrt{u^2+a^2})-\ln a$

方法二：令 $u=a\tan x$，$du=\frac{a}{\cos^2 x}dx$

被積函數 $\frac{u^2}{\sqrt{a^2+u^2}}=\frac{a^2\tan^2 x}{\sqrt{a^2(1+\tan^2 x)}}=\frac{a^2\tan^2 x}{a\sec x}$

$$\int\frac{u^2}{\sqrt{a^2+u^2}}du=\int\frac{a^2\sin^2 x}{\cos^2 x}\cdot\frac{\cos x}{1}\cdot\frac{a}{\cos^2 x}$$

$$=a^2\int\frac{\sin^2 x}{\cos^3 x}dx$$ （令 $t=\sin x$，$dt=\cos xdx$　代入）

$$=a^2\int\frac{t^2}{(1-t^2)^2}dt$$

$$=\frac{a^2}{4}\int -\frac{1}{1-t}-\frac{1}{1+t}+\frac{1}{(1-t)^2}+\frac{1}{(1+t)^2}dt$$

$$=\frac{a^2}{4}\left[\ln(1-t)-\ln(1+t)+\frac{1}{1-t}-\frac{1}{1+t}\right]+C'$$

$$=\frac{a^2}{4}\left[-\ln\frac{(1+t)}{(1-t)}+\frac{2t}{1-t^2}\right]+C'$$ （$t=\sin x=\frac{u}{\sqrt{a^2+u^2}}$ 代回式子）

$$=\frac{a^2}{4}\left[-\ln\left(\frac{1+\frac{u}{\sqrt{a^2+u^2}}}{1-\frac{u}{\sqrt{a^2+u^2}}}\right)+\frac{\frac{2u}{\sqrt{a^2+u^2}}}{\frac{a^2}{a^2+u^2}}\right]+C'$$

$$=-\frac{a^2}{4}\ln\frac{(\sqrt{a^2+u^2}+u)}{(\sqrt{a^2+u^2}-u)}\bullet\frac{(\sqrt{a^2+u^2}+u)}{(\sqrt{a^2+u^2}+u)}+\frac{a^2}{4}\frac{2u}{a^2}\sqrt{a^2+u^2}+C'$$

$$=-\frac{a^2}{4}\ln\left(\frac{\sqrt{a^2+u^2}+u}{a}\right)^2+\frac u2\sqrt{a^2+u^2}+C'$$

$$=-\frac{a^2}{2}\ln\left(\frac{\sqrt{a^2+u^2}+u}{a}\right)+\frac u2\sqrt{a^2+u^2}+C'$$

$$=-\frac{a^2}{2}\ln(\sqrt{a^2+u^2}+u)+\frac u2\sqrt{a^2+u^2}+C'-\frac{a^2}{2}\ln a$$

$$=-\frac{a^2}{2}\ln(\sqrt{a^2+u^2}+u)+\frac u2\sqrt{a^2+u^2}+C$$

【27】 $\int\frac{du}{u\sqrt{a^2+u^2}}=-\frac1a\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right|+C$

<解>：令 $u=a\tan x$

$$\int\frac{1}{a\tan x\cdot\frac{a}{\cos x}}\cdot\frac{a}{\cos^2 x}dx=\frac1a\int\frac{1}{\sin x}dx$$

$=\frac1a\int\csc x\,dx$（➲式【15】$\int\csc u\,du=\ln\left|\csc u-\cot u\right|+C$）

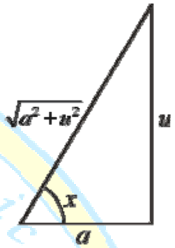
$$=\frac{1}{a}\ln\left|\csc u-\cot u\right|+C$$

$$=\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}}{u}-\frac{a}{u}\right|+C$$

$$=\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}-a}{u}\right|+C$$

$$=-\frac{1}{a}\ln\left|\frac{u}{\sqrt{a^2+u^2}-a}\cdot\frac{\sqrt{a^2+u^2}+a}{\sqrt{a^2+u^2}+a}\right|+C$$

$$=-\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right|+C$$



【28】$\int\frac{du}{u^2\sqrt{a^2+u^2}}=-\frac{\sqrt{a^2+u^2}}{a^2u}+C$

<解>：令 $u=a\tan x$ ， $du=\frac{a}{\cos^2 x}dx$

原式 $=\int\frac{1}{a^2\tan^2 x\cdot\frac{a}{\cos x}}\cdot\frac{a}{\cos^2 x}dx$

$$=\frac{1}{a^2}\int\frac{\cos x}{\sin^2 x}dx$$

$$=\frac{1}{a^2}\int\frac{1}{\sin^2 x}(d\sin x)$$

$$=\frac{1}{a^2}\cdot(-1)\cdot\frac{1}{\sin x}+C$$

$$=-\frac{\sqrt{a^2+u^2}}{a^2u}+C$$

【29】$\int\frac{du}{(a^2+u^2)^{\frac{3}{2}}}=\frac{u}{a^2\sqrt{a^2+u^2}}+C$

<解>：令 $u=a\tan x$ ， $du=a\frac{1}{\cos^2 x}dx$

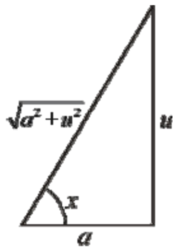
$$a^2+u^2=a^2+a^2\tan^2 x=\frac{a^2}{\cos^2 x}$$

$$\int\frac{\cos^3 x}{a^3}\cdot a\cdot\frac{1}{\cos^2 x}dx=\frac{1}{a^2}\int\cos x dx$$

$$=\frac{1}{a^2}\sin x+C$$

$$=\frac{1}{a^2}\cdot\frac{u}{\sqrt{a^2+u^2}}+C$$

$$=\frac{u}{a^2\sqrt{a^2+u^2}}+C$$



【提示】30-38 解題技巧

令 $u=a\sin x$

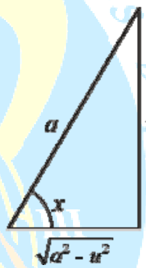
$du=a\cos x dx$

$\sqrt{a^2-u^2}=\sqrt{a^2-a^2\sin^2 x}=a\cos x$

$x=\sin^{-1}\frac{u}{a}$

$\cos x=\frac{\sqrt{a^2-u^2}}{a}$

$\sin x=\frac{u}{a}$



$\sin 2x=2\sin x\cos x=2\frac{u}{a}\cdot\frac{\sqrt{a^2-u^2}}{a}$

$\sin 4x=2\sin 2x\cos 2x$

$$=4\sin x\cos x(\cos^2 x-\sin^2 x)$$

$$=4\frac{u}{a}\cdot\frac{\sqrt{a^2-u^2}}{a}\left(\frac{a^2-u^2}{a^2}-\frac{u^2}{a^2}\right)$$

$$=\frac{4u}{a^4}\sqrt{a^2-u^2}\left(a^2-2u^2\right)$$

【30】$\int\sqrt{a^2-u^2}du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\frac{u}{a}+C$

<解>： $\int\sqrt{a^2-u^2}du=a^2\int\cos^2 x dx$

$$=a^2\int\frac{1+\cos 2x}{2}dx$$

$$=a^2\left(\frac{1}{2}x+\frac{1}{2}\cdot\frac{1}{2}\sin 2x\right)+C$$

$$=\frac{a^2}{2}\sin^{-1}\frac{u}{a}+\frac{u}{2}\sqrt{a^2-u^2}+C$$

（$\sin 2x=2\sin x\cos x=2\frac{u}{a}\cdot\frac{\sqrt{a^2-u^2}}{a}$ 代入）

【31】$\int u^2\sqrt{a^2-u^2}\,du=\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C$

<解>：原式$=\int a^2\sin^2 x\cdot a^2\cos^2 x\,dx$

$$=a^4\int\frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}dx$$

$$=\frac{a^4}{4}\int 1-\cos^2 2x\,dx$$

$$=\frac{a^4}{4}\int 1-\frac{1+\cos 4x}{2}dx$$

$$=\frac{a^4}{4}\left(\frac{1}{2}x-\frac{1}{2}\cdot\frac{1}{4}\sin 4x\right)+C$$

$$=\frac{a^4}{8}\sin^{-1}\frac{u}{a}+\frac{u}{8}\sqrt{a^2-u^2}(2u^2-a^2)+C$$

$\sin 4x=2\sin 2x\cos 2x$

$$=4\sin x\cos x(\cos^2 x-\sin^2 x)$$

$$=4\frac{u}{a}\cdot\frac{\sqrt{a^2-u^2}}{a}\left(\frac{a^2-u^2}{a^2}-\frac{u^2}{a^2}\right)$$

$$=\frac{4u}{a^4}\sqrt{a^2-u^2}\left(a^2-2u^2\right)$$

【32】$\int\frac{\sqrt{a^2-u^2}}{u}du=\sqrt{a^2-u^2}-a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C$

<解>：$\int\frac{a^2\cos^2 x}{a\sin x}dx=-a\int\frac{t^2}{1-t^2}dt$　　（令 $t=\cos x$，$dt=-\sin x\,dx$）

$$=-a\int\frac{t^2}{1-t^2}dt$$

$$=-a\int -1+\frac{1}{2(1-t)}+\frac{1}{2(1+t)}dt$$

$$=at-\frac{a}{2}\ln\left|\frac{1+t}{1-t}\right|+c$$

$$=\sqrt{a^2-u^2}-a\ln\left(\frac{a+\sqrt{a^2-u^2}}{u}\right)+C$$

$$\frac{1+t}{1-t}=\frac{\sqrt{a^2-u^2}+a}{a-\sqrt{a^2-u^2}}\cdot\frac{a+\sqrt{a^2-u^2}}{a+\sqrt{a^2-u^2}}=\left(\frac{a+\sqrt{a^2-u^2}}{u}\right)^2$$

【33】$\int\frac{\sqrt{a^2-u^2}}{u^2}du=-\frac{\sqrt{a^2-u^2}}{u}-\sin^{-1}\frac{u}{a}+C$

<解>：令 $u=a\sin x$，$du=a\cos x\,dx$ 代入

$$\int\frac{a^2\cos^2 x}{a^2\sin^2 x}dx=\int\frac{1+\cos 2x}{1-\cos 2x}dx$$

$$=\int -1+\frac{2}{1-\cos 2x}dx$$

$$=-x+\int\frac{1}{\sin^2 x}dx+C\qquad\text{（式 9）}$$

$$=-x-\cot x+C$$

$$=-\sin^{-1}\frac{u}{a}-\frac{\sqrt{a^2-u^2}}{u}+C$$

令 $t=\tan x$

$dt=\frac{1}{\cos^2 x}dx=(1+t^2)dx$

$\cos 2x=\frac{1-t^2}{1+t^2}$

$\int\frac{2}{1-\cos 2x}dx=\int\frac{1}{t^2}dt$

$=-t^{-1}+C$

$=-\cot x+C$

-1
$-\cos 2x+1\,)\;\cos 2x+1$
$-(\cos 2x-1)$
2

【34】 $\int \frac{u^2 du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

<解>：原式 $= \int \frac{a^2(\sin x)^2 a(\cos x)dx}{a\cos x}$

$= a^2 \int \frac{1-\cos 2x}{2}dx$

$= a^2\left(\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x\right) + C$　（ $\sin 2x = 2\sin x\cos x = 2\frac{u}{a}\cdot\frac{\sqrt{a^2-u^2}}{a}$ ）

$= \frac{a^2}{2}\sin^{-1}\frac{u}{a} - \frac{u}{2}\sqrt{a^2-u^2} + C$

【35】 $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

<解>：原式 $= \int \frac{a\cos x dx}{a\sin x \cdot a\cos x dx}$

$= \frac{1}{a}\int \csc x dx$　　➲ 式【15】 $\int \csc u du = \ln|\csc u - \cot u| + C$

$= \frac{1}{a}\ln|\csc x - \cot x| + C$

$= \frac{1}{a}\ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C$

$= -\frac{1}{a}\ln\left|\frac{u}{a-\sqrt{a^2-u^2}}\cdot\frac{a+\sqrt{a^2-u^2}}{a+\sqrt{a^2-u^2}}\right| + C$

$= -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

【36】 $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2 u}\cdot\sqrt{a^2-u^2} + C$

<解>：原式 $= \int \frac{a\cos x dx}{a^2\sin^2 x \cdot a\cos x}$

$= \frac{1}{a^2}\int \frac{1}{\sin^2 x}dx$　　➲ 【9】 $\int \csc^2 u du = -\cot u + C$

$= -\frac{1}{a^2}\cot x + C$

$= -\frac{1}{a^2}\cdot\frac{\sqrt{a^2-u^2}}{u} + C$

【37】 $\int (a^2-u^2)^{3/2} du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3}{8}a^4\sin^{-1}\frac{u}{a} + C$

<解>： $\int (a^2-u^2)^{3/2} du = \int a^3\cos^3 x \cdot a\cos x dx$

$= a^4\int (\frac{1+\cos 2x}{2})^2 dx$

$= \frac{a^4}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}dx$

$= \frac{a^4}{4}(x + 2\cdot\frac{1}{2}\sin 2x + \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{4}\sin 4x) + C$

$= \frac{3}{8}a^4\sin^{-1}\frac{u}{a} + \frac{ua^2}{2}\sqrt{a^2-u^2} + \frac{1}{8}u\sqrt{a^2-u^2}(a^2-2u^2) + C$

$= -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3}{8}a^4\sin^{-1}\frac{u}{a} + C$

【38】 $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

<解>：原式 $= \int \frac{a\cos x dx}{a^3\cos^3 x}$

$= \frac{1}{a^2}\int \sec^2 x dx$　　➲ 式【8】 $\int \sec^2 u du = \tan u + C$

$= \frac{1}{a^2}\tan x + C$

$= \frac{u}{a^2\sqrt{a^2-u^2}} + C$

【提示】39-40 解題技巧

令 $u = a\cosh x$ ， $du = a\sinh x dx$

$\cosh^2 x - \sinh^2 x = 1 \Rightarrow \cosh^2 x - 1 = \sinh^2 x$

$\cosh x = \frac{e^x + e^{-x}}{2}$　　　$(\cosh x)' = \sinh x$

$\sinh x = \dfrac{e^{x}-e^{-x}}{2}$ $\qquad (\sinh x)' = \cosh x$

因爲 $u = a\cosh x$

則 $\cosh x = \dfrac{u}{a} = \dfrac{e^{x}+e^{-x}}{2} \Rightarrow e^{x} = \dfrac{u+\sqrt{u^2-a^2}}{a}$

$\Rightarrow x = \ln\left|u+\sqrt{u^2-a^2}\right| - \ln|a|$

$\sinh 2x = 2\sinh x\cosh x = 2\sqrt{(\frac{u}{a})^2-1}\cdot(\frac{u}{a}) = 2\cdot\dfrac{u}{a^2}\sqrt{u^2-a^2}$

【39】 $\int\sqrt{u^2-a^2}\,du = \dfrac{u}{2}\sqrt{u^2-a^2} - \dfrac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

<解>：令 $u = a\cosh x$，$du = a\sinh x dx$

原式 $= a^2\int\sinh^2 x dx$

$= \dfrac{a^2}{4}\int\left(e^{2x}+e^{-2x}-2\right)dx$

$= \dfrac{a^2}{4}\left(\dfrac{e^{2x}-e^{-2x}}{2} - 2x\right) + c'$

$\dfrac{e^{2x}-e^{-2x}}{2} = \sinh 2x$

$= 2\sinh x\cosh x$

$= 2\cdot\sqrt{(\frac{u}{a})^2-1}\cdot\dfrac{u}{a}$

$= \dfrac{2u\sqrt{u^2-a^2}}{a^2}$

$= \dfrac{u}{2}\sqrt{u^2-a^2} - \dfrac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + \dfrac{a^2}{2}\ln|a| + c'$

$= \dfrac{u}{2}\sqrt{u^2-a^2} - \dfrac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + c$

【40】 $\int u^2\sqrt{u^2-a^2}\,du = \dfrac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \dfrac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

<解>：原式 $= a^4\int\sinh^2 x\cosh^2 xdx$

$= a^4\int(\dfrac{\sinh 2x}{2})^2 dx$

$= \dfrac{a^4}{16}\int(\dfrac{e^{2x}-e^{-2x}}{4})^2 dx$

$= \dfrac{a^4}{16}\int e^{4x}+e^{-4x}-2dx$

$= \dfrac{a^4}{16}\cdot(\dfrac{e^{4x}-e^{-4x}}{4}-2x)+c$

$= \dfrac{a^4}{16}\cdot(\dfrac{\sinh 4x}{2}-2x)+c$

$= \dfrac{1}{8}u(2u^2-a^2)\sqrt{u^2-a^2} - \dfrac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + c'$

$\sinh 4x = 2\sinh 2x\cosh 2x$

$= 4\sinh x\cosh x(\cosh^2 x-1)$

$= 4\dfrac{\sqrt{u^2-a^2}}{a}\cdot\dfrac{u}{a}\cdot\left(\dfrac{2u^2-a^2}{a^2}\right)$

$= \dfrac{4u\left(2u^2-a^2\right)}{a^4}\sqrt{u^2-a^2}$

【提示】解題技巧

令 $u = \dfrac{a}{\cos x}$，$du = \dfrac{a\sin x}{\cos^2 x}dx$

$\sqrt{u^2-a^2} = a\tan x = \dfrac{a\sin x}{\cos x}$

【41】 $\int\dfrac{\sqrt{u^2-a^2}}{u}du = \sqrt{u^2-a^2} - a\cos^{-1}\dfrac{a}{|u|} + c$

<解>：令 $u = \dfrac{a}{\cos x}$，$du = \dfrac{a\sin x}{\cos^2 x}dx$

$\int a\tan x\cdot\dfrac{\cos x}{a}\cdot\dfrac{a\sin x}{\cos^2 x}dx = a\int\dfrac{\sin^2 x}{\cos^2 x}dx$

$= a\int\dfrac{1-\cos 2x}{1+\cos 2x}dx$

$= a\int -1+\dfrac{2}{1+\cos 2x}dx$

$$= a\int -1+\frac{1}{\cos^2 x}dx \qquad \text{查【8】} \int \sec^2 u du = \tan u + C$$

$$= -ax + \tan x + c$$

$$= -a\cos^{-1}\frac{a}{u} + \frac{a}{a}\sqrt{u^2-a^2} + c$$

【42】 $\int \frac{\sqrt{u^2-a^2}}{u^2}du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

<解>：令 $u=\frac{a}{\cos x}$ ， $du=\frac{a\sin x}{\cos^2 x}dx$

$$\sqrt{u^2-a^2} = a\tan x = \frac{a\sin x}{\cos x}$$

$$\text{原式} = \int \frac{\sin^2 x}{\cos x}dx \qquad (\text{令 } t=\sin x \text{ ， } dt=\cos x dx)$$

$$= \int \frac{t^2}{1-t^2}dt$$

$$= \int -1 + \frac{1}{2(1-t)} + \frac{1}{2(1+t)}dt$$

$$= -t + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c'$$

$$= -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a + c'$$

$$= -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + c$$

$$\left[\frac{1+t}{1-t} = \frac{1+\frac{\sqrt{u^2-a^2}}{u}}{1-\frac{\sqrt{u^2-a^2}}{u}} \cdot \frac{u+\sqrt{u^2-a^2}}{u+\sqrt{u^2-a^2}} = \left(\frac{u+\sqrt{u^2-a^2}}{a}\right)^2\right]$$

【43】 $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

<解>：令 $u=\frac{a}{\cos x}$ ， $du=\frac{a\sin x}{\cos^2 x}dx$

$$\sqrt{u^2-a^2} = a\tan x = \frac{a\sin x}{\cos x}$$

$$\text{原式} = \int \frac{\cos x}{a\sin x}\cdot\frac{a\sin x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos x}dx \qquad \text{查詢式【14】} \int \sec u du = \ln\left|\sec u + \tan u\right| + C$$

$$= \ln\left|\sec x + \tan x\right| + c'$$

$$= \ln\left|\frac{u+\sqrt{u^2-a^2}}{a}\right| + c'$$

$$= \ln\left|u+\sqrt{u^2-a^2}\right| - \ln a + c'$$

$$= \ln\left|u+\sqrt{u^2-a^2}\right| + c$$

【44】 $\int \frac{u^2 du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

<解>：令 $u=\frac{a}{\cos x}$ ， $du=\frac{a\sin x}{\cos^2 x}dx$

$$\sqrt{u^2-a^2} = a\tan x = \frac{a\sin x}{\cos x}$$

$$\text{原式} = \int \frac{a^2}{\cos^2 x}\cdot\frac{\cos x}{a\sin x}\cdot\frac{a\sin x}{\cos^2 x}dx$$

$$= a^2\int \frac{1}{\cos^3 x}dx$$

$$= a^2\int \frac{1}{(1-\sin^2 x)\cos x}dx \qquad (\text{令 } t=\sin x \text{ ， } dt=\cos x dx)$$

$$= a^2\int \frac{1}{(1-t^2)^2}dt \qquad \left[\frac{1}{(1-t^2)^2} = \frac{1}{4(1-t)} + \frac{1}{4(1-t)^2} + \frac{1}{4(1+t)} + \frac{1}{4(1+t)^2}\right]$$

$$= \frac{a^2}{4}\int \frac{1}{(1-t)} + \frac{1}{(1-t)^2} + \frac{1}{(1+t)} + \frac{1}{(1+t)^2}dt$$

$$= \frac{a^2}{4}\left(-\ln|1-t| + \frac{1}{1-t} + \ln|1+t| - \frac{1}{1+t}\right) + c$$

$$= \frac{a^2}{4}\left(\ln\left|\frac{1+t}{1-t}\right| + \frac{2t}{1-t^2}\right) + c'$$

$$\frac{1+t}{1-t} = \left(\frac{u+\sqrt{u^2-a^2}}{a}\right)^2$$

$$\frac{2t}{1-t^2}=\frac{2\cdot\frac{\sqrt{u^2-a^2}}{u}}{1-\frac{u^2-a^2}{u^2}}=\frac{2u\sqrt{u\ \ -a^2}}{u^2-u^2+a^2}=\frac{2u\sqrt{u^2-a^2}}{a^2}$$

$$=\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|-\frac{a^2}{2}\ln|a|+\frac{u\sqrt{u^2-a^2}}{2}+c'$$

$$=\frac{u}{2}\sqrt{u^2-a^2}+\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+c$$

【45】 $\int\frac{du}{u^2\sqrt{u^2-a^2}}=\frac{\sqrt{u^2-a^2}}{a^2u}+C$

<解>：令 $u=\frac{a}{\cos x}$，$du=\frac{a\sin x}{\cos^2 x}dx$

$$\sqrt{u^2-a^2}=a\tan x=\frac{a\sin x}{\cos x}$$

$$原式=\int\frac{\cos^2 x}{a^2}\cdot\frac{\cos x}{a\sin x}\cdot\frac{a\sin x}{\cos^2 x}dx$$

$$=\frac{1}{a^2}\int\cos x dx$$

$$=\frac{1}{a^2}\sin x+c$$

$$=\frac{\sqrt{u^2-a^2}}{a^2u}+c$$

【46】 $\int\frac{du}{(u^2-a^2)^{3/2}}=-\frac{u}{a^2\sqrt{u^2-a^2}}+C$

<解>：令 $u=\frac{a}{\cos x}$，$du=\frac{a\sin x}{\cos^2 x}dx$

$$\sqrt{u^2-a^2}=a\tan x=\frac{a\sin x}{\cos x}$$

$$原式=\int\frac{1}{a^3\tan^3 x}du$$

$$=\int\frac{\cos^3 x}{a^3\cdot\sin^3 x}\cdot\frac{a\sin x}{\cos^2 x}dx$$

$$=\frac{1}{a^2}\int\frac{\cos x}{\sin^2 x}dx$$ （令 $t=\sin x$，$dt=\cos x dx$）

$$=\frac{1}{a^2}\int\frac{1}{t^2}dt$$

$$=-\frac{1}{a^2}\frac{1}{t}+c$$

$$=-\frac{u}{a^2\sqrt{u^2-a^2}}+c$$

【提示】47-56 解題技巧

令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

【47】 $\int\frac{udu}{a+bu}=\frac{1}{b^2}(a+bu-a\ln|a+bu|)+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

$$原式=\int\frac{t-a}{b}\cdot\frac{1}{t}\cdot\frac{dt}{b}$$

$$=\frac{1}{b^2}\int 1-\frac{a}{t}dt$$

$$=\frac{1}{b^2}(t-a\ln t)+c$$

$$=\frac{1}{b^2}(a+bu-a\ln|a+bu|)+c$$

【48】 $\int\frac{u^2du}{a+bu}=\frac{1}{2b^3}[(a+bu)^2-4a(a+bu)+2a^2\ln|a+bu|]+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

$$原式=\int\frac{(t-a)^2}{b^2}\cdot\frac{1}{t}\cdot\frac{dt}{b}$$

$$=\frac{1}{b^3}\int t-2a+\frac{a^2}{t}dt$$

$$=\frac{1}{2b^3}(t^2-4at+2a^2\ln t)+c$$

$$=\frac{1}{2b^3}[(a+bu)^2-4a(a+bu)+2a^2\ln|a+bu|]+c$$

【49】 $\int\frac{du}{u(a+bu)}=\frac{1}{a}\ln\left|\frac{u}{a+bu}\right|+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{b}{t-a}\cdot\frac{1}{t}\cdot\frac{dt}{b}$ $\qquad \frac{1}{t-a}\cdot\frac{1}{t}=+\frac{1}{a}\frac{1}{(t-a)}-\frac{1}{at}$

$=\frac{1}{a}\int\frac{1}{t-a}-\frac{1}{t}dt$

$=\frac{1}{a}[\ln|t-a|-\ln|t|]+c'$ $\qquad \ln|t-a|=\ln bu=\ln u-\ln b$

$=\frac{1}{a}(\ln|u|-\ln|a+bu|-\ln b)+c'$ $\qquad -\ln b+c'=c$

$=\frac{1}{a}\ln\left|\frac{u}{a+bu}\right|+c$

【50】 $\int\frac{du}{u^2(a+bu)}=-\frac{1}{au}+\frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right|+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{b^2}{(t-a)^2}\cdot\frac{1}{t}\cdot\frac{dt}{b}$

$=b\int\frac{A}{t}+\frac{B}{t-a}+\frac{C}{(t-a)^2}dt$ ……….(1)

(1)式之分子： $A(t-a)^2+Bt(t-a)+Ct=1$

令 $t=a \Rightarrow aC=1 \Rightarrow C=\frac{1}{a}$

令 $t=0 \Rightarrow Aa^2=1 \Rightarrow A=\frac{1}{a^2}$

比較 t 之係數： $-2aA-aB+C=0 \Rightarrow B=\frac{-1}{a^2}$

$=\frac{b}{a^2}\int\frac{1}{t}-\frac{1}{t-a}+\frac{a}{(t-a)^2}dt$

$=\frac{b}{a^2}(\ln|t|-\ln|t-a|-\frac{a}{t-a})+c'$ $\qquad \ln|t-a|=\ln|b|+\ln|u|$

$=\frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right|-\frac{1}{au}+c$

【51】 $\int\frac{udu}{(a+bu)^2}=\frac{a}{b^2(a+bu)}+\frac{1}{b^2}\ln|a+bu|+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{t-a}{b}\cdot\frac{1}{t^2}\cdot\frac{dt}{b}$

$=\frac{1}{b^2}\int\frac{1}{t}-\frac{a}{t^2}dt$

$=\frac{1}{b^2}(\ln|t|+\frac{a}{t})+c$

$=\frac{1}{b^2}\ln|a+bu|+\frac{a}{b^2(a+bu)}+c$

【52】 $\int\frac{du}{u(a+bu)^2}=\frac{1}{a(a+bu)}-\frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right|+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{b}{t-a}\cdot\frac{1}{t^2}\cdot\frac{dt}{b}$

$=\int\frac{1}{t^2(t-a)}dt$

$=\int\left(\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t-a}\right)dt$

$\frac{1}{t^2(t-a)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t-a}$

分子 $= At(t-a)+B(t-a)+Ct^2=1$

令 $t=0 \Rightarrow -Ba=1 \Rightarrow B=\frac{-1}{a}$

令 $t=a \Rightarrow Ca^2=1 \Rightarrow C=\frac{1}{a^2}$

t 的係數：$-aA+B=0 \Rightarrow A=\frac{-1}{a^2}$

$$=-\frac{1}{a^2}\int\frac{1}{t}+\frac{a}{t^2}-\frac{1}{t-a}dt$$

$$=-\frac{1}{a^2}(\ln t-\frac{a}{t}-\ln|t-a|)+c'$$

$$=-\frac{1}{a^2}(\ln|a+bu|-\frac{a}{a+bu}-\ln|u|-\ln|b|)+c' \qquad -\ln|b|+c'=c$$

$$=-\frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right|+\frac{1}{a(a+bu)}+c$$

【53】$\int\frac{u^2du}{(a+bu)^2}=\frac{1}{b^3}(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|)+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{(t-a)^2}{b^2}\cdot\frac{1}{t^2}\cdot\frac{dt}{b}$

$$=\frac{1}{b^3}\int 1-\frac{2a}{t}+\frac{a^2}{t^2}dt$$

$$=\frac{1}{b^3}(t-2a\ln t-\frac{a^2}{t})+c$$

$$=\frac{1}{b^3}(a+bu-2a\ln|a+bu|-\frac{a^2}{a+bu})+c$$

【54】$\int u\sqrt{u+bu}du=\frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{t-a}{b}\cdot t^{1/2}\cdot\frac{dt}{b}$

$$=\frac{1}{b^3}\int t^{3/2}-at^{1/2}dt$$

$$=\frac{1}{b^2}(\frac{2}{5}t^{5/2}-\frac{2}{3}at^{3/2})+c$$

$$=\frac{2}{15b^2}t^{3/2}(3t-5a)+c$$

$$=\frac{2}{15b^2}(a+bu)^{3/2}(3bu-2a)+c$$

【55】$\int\frac{udu}{\sqrt{a+bu}}=\frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{t-a}{b}\cdot t^{-1/2}\cdot\frac{dt}{b}$

$$=\frac{1}{b^2}\int t^{1/2}-at^{-1/2}dt$$

$$=\frac{1}{b^2}(\frac{2}{3}t^{3/2}-2at^{1/2})+c$$

$$=\frac{2}{3b^2}t^{1/2}(t-3a)+c$$

$$=\frac{2}{3b^2}\sqrt{a+bu}(bu-2a)+c$$

【56】$\int\frac{u^2du}{\sqrt{a+bu}}=\frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu}+c$

<解>：令 $t=a+bu \Rightarrow u=\frac{t-a}{b}$

$dt=bdu \Rightarrow du=\frac{dt}{b}$

原式 $=\int\frac{(t-a)^2}{b^2}\cdot t^{-1/2}\cdot\frac{dt}{b}$

$$=\frac{1}{b^3}\int t^{3/2}-2at^{1/2}+a^2t^{-1/2}dt$$

$$=\frac{1}{b^3}\left(\frac{2}{5}t^{5/2}-\frac{4}{3}at^{3/2}+2a^2t^{1/2}\right)+c$$

$$=\frac{2}{15b^3}t^{1/2}(3t^2-10at+15a^2)+c$$

$$=\frac{2}{15b^3}\sqrt{a+bu}\,(8a^2+3b^2u^2-4abu)+c$$

【57】 $\displaystyle\int\frac{du}{u\sqrt{a+bu}}=\frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+c \qquad \text{if} \quad a>0$

$$=\frac{1}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}}+c \qquad \text{if} \quad a<0$$

<解>：若 $a>0$ ➲ 令 $t^2=a+bu \Rightarrow u=\dfrac{t^2-a}{b}$

$$du=\frac{2tdt}{b}$$

$$原式=\int\frac{b}{t^2-a}\cdot\frac{1}{t}\cdot\frac{2tdt}{b}$$

$$=2\int\frac{1}{t^2-a}dt$$

$$\frac{1}{t^2-a}=\frac{A}{t+\sqrt{a}}+\frac{B}{t-\sqrt{a}}$$

分子 $=A(t-\sqrt{a})+B(t+\sqrt{a})=1$

t 的係數：$A+B=0 \qquad \Rightarrow \quad A=-\dfrac{1}{2\sqrt{a}}$

t^0 的係數：$-A\sqrt{a}+B\sqrt{a}=1 \qquad B=\dfrac{1}{2\sqrt{a}}$

$$=\frac{1}{\sqrt{a}}\int-\frac{1}{t+\sqrt{a}}+\frac{1}{t-\sqrt{a}}dt$$

$$=\frac{1}{\sqrt{a}}\ln\left|\frac{t-\sqrt{a}}{t+\sqrt{a}}\right|+c$$

$$=\frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+c$$

若 $a<0$ ➲ 令 $\tan t=\sqrt{\dfrac{a+bu}{-a}} \Rightarrow \tan^2 t=\dfrac{a+bu}{-a} \Rightarrow u=\dfrac{-a(\tan^2 t+1)}{b}$

$$du=\frac{-a}{b}\cdot 2\tan t\cdot\frac{1}{\cos^2 t}dt$$

$$原式=\int\frac{b}{-a(\tan^2 t+1)}\cdot\frac{1}{\sqrt{-a}\tan t}\cdot\frac{-a}{b}\cdot 2\tan t\cdot\frac{1}{\cos^2 t}dt$$

$$=\frac{2}{\sqrt{-a}}\int 1dt$$

$$=\frac{2}{\sqrt{-a}}t+c$$

$$=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}}+c$$

【58】 $\displaystyle\int\frac{\sqrt{a+bu}}{u}du=2\sqrt{a+bu}+a\int\frac{du}{u\sqrt{a+bu}}$

<解>：$t^2=a+bu \Rightarrow u=\dfrac{t^2-a}{b}$

$2tdt=bdu \Rightarrow du=\dfrac{2t}{b}dt$

$$原式=\int\frac{b}{t^2-a}\cdot t\cdot\frac{2t}{b}dt$$

$$=2\int 1+\frac{a}{t^2-a}dt$$

$$=2t+2a\int\frac{1}{t^2-a}dt+c$$

由【57】可知 $\displaystyle 2\int\frac{1}{t^2-a}dt=\int\frac{du}{u\sqrt{a+bu}}$

$$=2\sqrt{a+bu}+2a\int\frac{1}{bu}\cdot\frac{bdu}{2\sqrt{a+bu}}+c$$

$$=2\sqrt{a+bu}+a\int\frac{du}{u\sqrt{a+bu}}$$

【59】 $\displaystyle\int\frac{\sqrt{a+bu}}{u^2}du=-\frac{\sqrt{a+bu}}{u}+\frac{b}{2}\int\frac{du}{u\sqrt{a+bu}}$

<解>：$\displaystyle\int w'vdu=wv-\int wv'du$

令 $v=\sqrt{a+bu} \Rightarrow v'=\dfrac{b}{2}(a+bu)^{-1/2}$

$w'=u^{-2} \Rightarrow w=-u^{-1}$

代入得到 $\displaystyle\int\frac{\sqrt{a+bu}}{u^2}du=-\frac{\sqrt{a+bu}}{u}+\frac{b}{2}\int\frac{du}{u\sqrt{a+bu}}$

【60】 $\int u^n\sqrt{a+bu}\,du=\frac{2}{b(2n+3)}[u^n(a+bu)^{3/2}-na\int u^{n-1}\sqrt{a+bu}\,du]$

<解>：

【61】 $\int\frac{u^n du}{\sqrt{a+bu}}=\frac{2u^n\sqrt{a+bu}}{b(2n+1)}-\frac{2na}{b(2n+1)}\int\frac{u^{n-1}du}{\sqrt{a+bu}}$

<解>：

【62】 $\int\frac{du}{u^n\sqrt{a+bu}}=-\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}}-\frac{b(2n-3)}{2a(n-1)}\int\frac{du}{u^{n-1}\sqrt{a+bu}}$

<解>：

【63】 $\int\sin^2 u\,du=\frac{1}{2}u-\frac{1}{4}\sin 2u+C$

<解>： $\int\sin^2 u\,du=\int\frac{1-\cos 2u}{2}du$

$=\frac{1}{2}u-\frac{1}{4}\sin 2u+c$

【64】 $\int\cos^2 u\,du=\frac{1}{2}u+\frac{1}{4}\sin 2u+C$

<解>： $\int\cos^2 u\,du=\int\frac{1+\cos 2u}{2}du$

$=\frac{1}{2}u+\frac{1}{4}\sin 2u+c$

【65】 $\int\tan^2 u\,du=\tan u-u+C$

<解>： $\int\tan^2 u\,du=\int\frac{\sin^2 u}{\cos^2 u}du$

$=\int\frac{1-\cos 2u}{1+\cos 2u}du$

$=\int -1+\frac{2}{1+\cos 2u}du$ 　　查【8】 $\frac{2}{1+\cos 2u}=\frac{1}{\cos^2 u}=\sec^2 u$

$=-u+\tan u+c$ 　　$\Rightarrow\ \int\sec^2 u\,du\ =\ \tan u+C$

【66】 $\int\cot^2 u\,du=-\cot u-u+C$

<解>： $\int\cot^2 u\,du=\int\frac{\sin^2 u}{\cos^2 u}du$

$=\int\frac{1-\cos 2u}{1+\cos 2u}du$

$=\int -1+\frac{2}{1+\cos 2u}du$

$=\int -1+\csc^2 u\,du$

$=-u-\cot u+c$ 　　查【9】 $\int\csc^2 u\,du\ =-\cot u+C$

【67】 $\int\sin^3 u\,du=-\frac{1}{3}(2+\sin^2 u)\cos u+C$

<解>： $\int\sin^2 u\,du=-\int(1-\cos^2 u)d\cos u$

$=-\cos u+\frac{1}{3}(\cos u)^3+c$

$=\frac{\cos u}{3}(-3+1-\sin^2 u)+c$

$=\frac{-1}{3}(2+\sin^2 u)\cos u+c$

【68】 $\int\cos^3 u\,du=\frac{1}{3}(2+\cos^2 u)\sin u+C$

<解>： $\int\cos^3 u\,du=\int 1-\sin^2 u(d\sin u)$

$=\sin u-\frac{1}{3}\sin^3 u+c$

$=\frac{1}{3}\sin u(3-1+\cos^2 u)+c$

$=\frac{1}{3}(2\cos u)\sin u+c$

【69】 $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

<解>： $\int \tan^3 u\,du = \int \frac{\sin^3 u}{\cos^3 u}du$

$$= -\int \frac{\sin^2 u}{\cos^2 u}(d\cos u)$$

$$= -\int \frac{1}{\cos^3 u} - \frac{1}{\cos u}(d\cos u)$$

$$= \frac{1}{2\cos^2 u} + \ln|\cos u| + c'$$

$$= \frac{1}{2}(1+\tan^2 u) + \ln|\cos u| + c'$$

$$= \frac{1}{2}\tan^2 u + \ln|\cos u| + c$$

【70】 $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

<解>： $\int \cot^3 u\,du = \int \frac{\cos^3 u}{\sin^3 u}du$

$$= \int \frac{1-\sin^2 u}{\sin^3 u}(d\sin u)$$

$$= -\frac{1}{2}\frac{1}{\sin^2 u} - \ln|\sin u| + c$$

$$= -\frac{1}{2}(1+\cot^2 u) - \ln|\sin u| + c$$

$$= -\frac{1}{2}\cot^2 u - \ln|\sin u| + c$$

【71】 $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

<解>： $\int \sec^3 u\,du = \int \frac{1}{\cos^3 u}du$　　　令 $t = \sin u$ ， $dt = \cos u\,du$

$$= \int \frac{1}{(1-t^2)}dt$$

$$= \frac{1}{4}\int \frac{1}{1-t} + \frac{1}{1+t} + \frac{1}{(1-t)^2} + \frac{1}{(1+t)^2}dt$$

$$= \frac{1}{4}(\ln\left|\frac{1+t}{1-t}\right| + \frac{1}{1-t} - \frac{1}{1+t}) + C$$

$$= \frac{1}{4}(\ln\left|\frac{1+\sin u}{1-\sin u}\right| + \frac{2\sin u}{1-\sin^2 u}) + C \qquad \left(\frac{2\sin u}{1-\sin^2 u} = \frac{2\sin u}{\cos^2 u} = 2\cdot\frac{1}{\cos u}\cdot\frac{\sin u}{\cos u}\right)$$

$$= \frac{1}{2}\ln|\sec u + \tan u| + \frac{1}{2}\sec u\tan u + C$$

$$\left(\ln\left|\frac{1+\sin u}{1-\sin u}\right| = \ln\left|\frac{(1+\sin u)^2}{1-\sin^2 u}\right| = \ln(\frac{1+\sin u}{\cos u})^2 = 2\ln|\sec u + \tan u|\right)$$

【72】 $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

<解>： $\int \csc^3 u\,du = \int \frac{1}{\sin^3 u}du$　　　令 $t = \cos u$ ， $dt = -\sin u\,du$

$$= -\int \frac{1}{(1-t)^2}dt$$

$$= -\frac{1}{4}\int \frac{1}{1-t} + \frac{1}{1+t} + \frac{1}{(1-t)^2} + \frac{1}{(1+t)^2}dt$$

$$= -\frac{1}{4}\left(\ln\left|\frac{1+\cos u}{1-\cos u}\right| + \frac{2\cos u}{1-\cos^2 u}\right) + C$$

$$\left(\frac{2\cos u}{1-\cos^2 u} = \frac{2\cos u}{\sin^2 u} = 2\cdot\frac{1}{\sin u}\cdot\frac{\cos u}{\sin u} = 2\cdot\csc u\cot u\right)$$

$$= \frac{1}{2}\ln|\csc u - \cot u| - \frac{1}{2}\csc u\cot u + C - \ln\left|\frac{1+\cos u}{1-\cos u}\right|$$

$$= \ln\left|\frac{1-\cos u}{1+\cos u}\right|$$

$$= \ln\frac{(1-\cos u)^2}{1-\cos^2 u}$$

$$= 2\ln(\frac{1-\cos u}{\sin u})$$

$$= 2\ln\left|\frac{1}{\sin u} - \frac{\cos u}{\sin u}\right|$$

$$= 2\ln|\csc u - \cot u|$$

$$\int wv'du = wv - \int w'v\,du$$

【73】$\int \sin^n u du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u du$

<解>：令 $w=\sin^{n-1}u$ ， $w'=(n-1)\sin^{n-2}u\cos u$

$v'=\sin u$ ， $v=-\cos u$

$$\int \sin^n u du = -\sin^{n-1}u\cos u - (n-1)\int \sin^{n-2}u\cos^2 u du$$

（右式 $\cos^2 u = 1-\sin^2 u$ 代入）

$$1-(n-1)\int \sin^n u du = -\sin^{n-1}u\cos u - (n-1)\int \sin^{n-2}u du$$

$$\int \sin^n u du = \frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u du$$

【74】$\int \cos^n u du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u du$

<解>：令 $w=\cos^{n-1}u$ $w'=-(n-1)\cos^{n-2}u\sin u$

$v'=\cos u$ $v=\sin u$

$$\int \cos^n u du = \cos^{n-1}u\sin u + (n-1)\int \cos^{n-2}u\sin^2 u du \quad (\sin^2 u = 1-\cos^2 u \text{ 代入})$$

$$1-(n-1)\int \cos^n u du = \cos^{n-1}u\sin u + (n-1)\int \cos^{n-2}u du$$

$$\int \cos^n u du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u du$$

【75】$\int \tan^n u du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u du$

<解>：令 $w=\sin^{n-1}u$ $w'=(n-1)\sin^{n-2}u\cos u$

$v'=\cos^{-n}u\sin u$ $v=\frac{1}{n-1}\cos^{-n+1}u$

$$\int \tan^n u du = \frac{1}{n-1}\tan^{n-1}u - \frac{n-1}{n-1}\int \sin^{n-2}u\cos^{-n+2}u du$$

$$= \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u du$$

【76】$\int \cot^n u du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u du$

<解>：令 $w=\cos^{n-1}u$ $w'=(1-n)\cos^{n-2}u\sin u$

$v'=\sin^{-n}u\cos u$ $v=\frac{1}{1-n}\sin^{-n+1}u$

$$\int \cot^n u du = -\frac{1}{n-1}\cos^{n-1}u\sin^{-n+1}u - \frac{1-n}{1-n}\int \cos^{n-2}u\sin^{-n+2}u du$$

$$= -\frac{1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u du$$

【77】$\int \sec^n u du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u du$

<解>：令 $w=\sec^{n-2}u$ $w'=(n-2)\sec^{n-3}u\cdot\sec u\tan u$

$v'=\sec^2 u = \frac{1}{\cos^2 u}$ $v=\tan u$

（ $\frac{d(\sec u)}{du}=\sec u\tan u$ ）

$$\int \sec^n u du = \tan u\sec^{n-2}u - (n-2)\int \sec^{n-2}u\tan^2 u du \quad (\tan^2 u = \sec^2 u - 1 \text{代入})$$

$$1+(n-2)\int \sec^n u du = \tan u\sec^{n-2}u + (n-2)\int \sec^{n-2}u du$$

$$\int \sec^n u du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u du$$

【78】$\int \csc^n u du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u du$

<解>：令 $w=\csc^{n-2}u$ $w'=(n-2)\csc^{n-3}u\cdot(-\csc u\cot u)$

$v'=\csc^2 u = \frac{1}{\sin^2 u}$ $v=-\cot u$

（ $\frac{d(\csc u)}{du}=-\csc u\cot u$ ）

$$\int \csc^n u du = -\cot u\csc^{n-2}u + (n-2)\int \csc^{n-2}u\cot^2 u du \quad (\cot^2 u = \csc^2 u - 1 \text{代入})$$

$$1+(n-2)\int \csc^n u du = -\cot u\csc^{n-2}u + (n-2)\int \csc^{n-2}u du$$

$$\int \csc^n u du = -\frac{1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u du$$

【79】$\int \sin au\sin bu du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

<解>：

【80】 $\int \cos au \cos bu du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

<解>：

【81】 $\int \sin au \cos bu du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

<解>：

【82】 $\int u \sin u du = \sin u - u\cos u + C$

<解>：令 $w = u$ $\quad w' = 1$

$v' = \sin u$ $\quad v = -\cos u$

$$\int u \sin u du = -u\cos u + \int \cos u du$$

$$= -u\cos u + \sin u + C$$

【83】 $\int u \cos u du = \cos u + u \sin u + C$

<解>：令 $w = u$ $\quad w' = 1$

$v' = \cos u$ $\quad v = \sin u$

$$\int u \cos u du = u \sin u - \int \sin u du$$

$$= u \sin u + \cos u + C$$

【84】 $\int u^n \sin u du = -u^n \cos u + n\int u^{n-1} \cos u du$

<解>：

【85】 $\int u^n \cos u du = u^n \sin u - n\int u^{n-1} \sin u du$

<解>：

【86】 $\int \sin^n u \cos^m u du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u du$

$$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u du$$

<解>：
